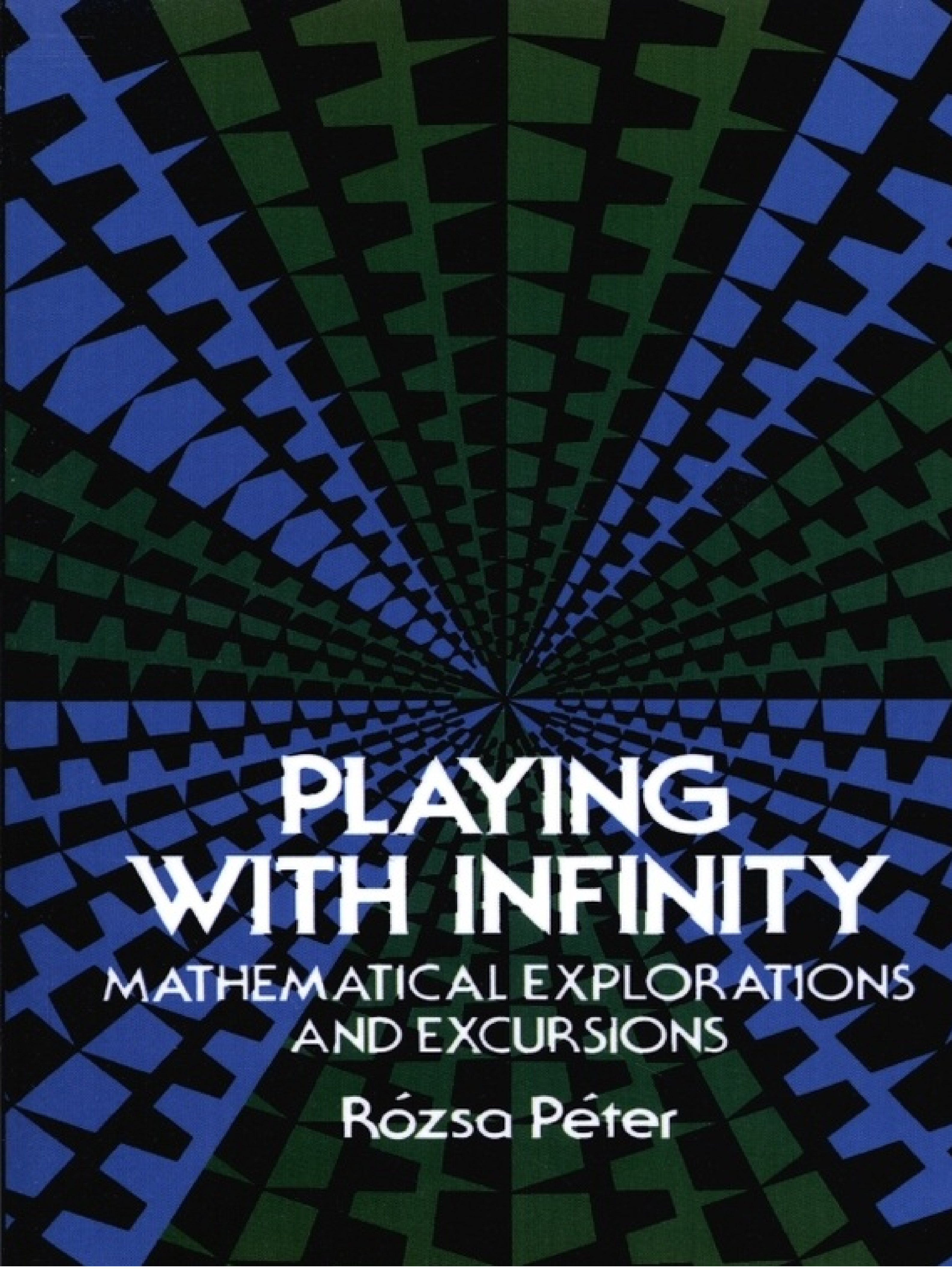
PLAYING
WITH INFINITY
MATHEMATICAL EXPLORATIONS
AND EXCURSIONS
Rózsa Péter

PLAYING WITH INFINITY

MATHEMATICAL EXPLORATIONS AND EXCURSIONS

BY

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TRANSLATED BY

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Dedicated to my brother, Dr. Nicolas Politzer, who perished at Colditz in Saxony, 1945

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PREFACE

THIS book is written for intellectually minded people who are not mathematicians. It is written for men of literature, of art, of the humanities. I have received a great deal from the arts and I would now like in my turn to present mathematics and let everyone see that mathematics and the arts are not so different from each other. I love mathematics not only for its technical applications, but principally because it is beautiful; because man has breathed his spirit of play into it, and because it has given him his greatest game–the encompassing of the infinite. Mathematics can give to the world such worthwhile things–about ideas, about infinity; and yet how essentially human it is–unlike the dull multiplication table, it bears on it for ever the stamp of man's handiwork.

The popular nature of the book does not mean that the subject is approached superficially. I have endeavoured to present concepts with complete clarity and purity so that some new light may have been thrown on the subject even for mathematicians and certainly for teachers. What has been left ou is the systematization which can so easily become boring; in other words, only technicalities have been omitted. (It is not the purpose of the book to teach anyone mathematical techniques.) If an interested pupil picks up this book it will give him a picture of the whole of mathematics. In the beginning I did not mean the book to be so full; the material expanded itself as I was writing it and th number of subjects which could be omitted rapidly decreased. If there were parts to which memories of boredom previously attached I felt that I was picking up some old piece of furniture and blowing the dust off in order to make it shine.

It is possible that the reader may find the style a little naive in places, but I do not mind this. A naive point of view in relation to simple facts always conjures up the excitement of new discovery.

I shall tell the reader in the Introduction how the book originated. The writer of whom I speak ther is Marcell Benedek. I began by writing to him about differentiation and it was his idea that a book could grow out of these letters.

I do not refer to any sources. I have learned a lot from others but today I can no longer say with certainty from whence each piece came. There was no book in front of me while I was writing. Here and there certain similes came to my mind with compelling force, the origins of which I could sometimes remember; for example, the beautiful book by Rademacher and Toeplitz,[1] or the excellen introduction to analysis by Beke.[2] Once a method had been formed in my mind I could not really wri it in any other way just to be more original. I chiefly refer, in this connexion, to the ideas I gained from László Kalmár. He was a contemporary of mine as well as my teacher in mathematics. Anythin I write is inseparably linked with his thoughts. I must mention, in particular, that the 'chocolate example', with the aid of which infinite series are discussed, originated with him, as well as the whol idea of the building up of logarithm tables.

I shall have to quote my little collaborators in the schoolroom by their christian names; they will surely recognize themselves. Here I must mention my pupil Kató, who has just finished the fourth year at the grammar school and contributed to the book while it was being written. It is to her that I must be grateful for being able to see the material with the eyes of a gifted pupil.

The most important help I received was from those people who have no mathematical interests. M dear friend Béla Lay, theatrical producer, who had always believed that he had no mathematical sens followed all the chapters as they were being written; I considered a chapter finished only when he wa satisfied with it. Without him the book perhaps would never have been written.

Pál Csillag examined the manuscript from the point of view of the mathematician; also László Kalmár found time, at the last minute, for a quick look. I am grateful to them for the certainty I feel that everything in the book is right.

Budapest

Autumn, 1943

RÓZSA PÉTER

PREFACE TO THE ENGLISH EDITION

SINCE 1943, seventeen eventful years have passed. During this time my mathematician friend, Pál Csillag, and my pupil, Kató (Kató Fuchs), have fallen victims of Fascism. The father of my pupil Anna, who suffered imprisonment for seventeen years for illegal working-class activity, has been freed. In this way perhaps even in Anna’s imagination the straight lines forever approaching one another will meet. (See page 218.) No book could appear during the German occupation; a lot of existing copies were destroyed by bombing, the remaining copies appeared in 1945–on the first free book-day.

I am very grateful to Dr. Emma Barton, who took up the matter of the English publication of my book, to Professor Dr. R. L. Goodstein, who brought it to a head, to Dr. Z. P. Dienes for the careful translation and to Messrs. G. Bell & Sons for making possible the propagation of the book in the English-speaking world.

The reader should remember that the book mirrors my methods of thinking as they were in 1943; I have hardly altered anything in it. Only the end has been altered substantially. Since then, László Kalmár and I have proved that the existence of absolutely undecidable problems follows from Gödel Theorem on relatively undecidable problems, but of course in no circumstances can a consequence b more important than the Theorem from which it follows.

Budapest
1960

RÓZSA PÉTE

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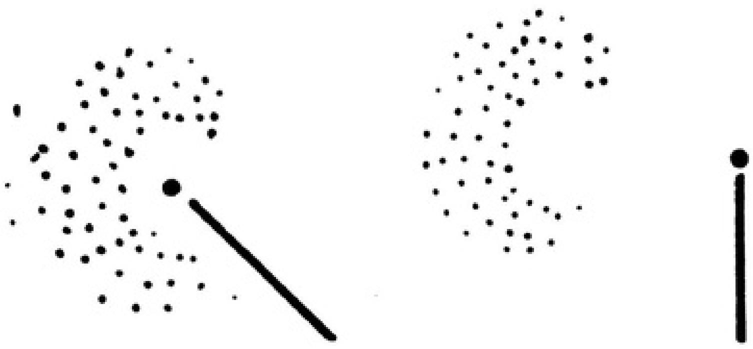
INTRODUCTION

A CONVERSATION I had a long time ago comes into my mind. One of our writers, a dear friend of mine, was complaining to me that he felt his education had been neglected in one important aspect, namely he did not know any mathematics. He felt this lack while working on his own ground, while writing. He still remembered the co-ordinate system from his school mathematics, and he had alread used this in similes and imagery. He felt that there must be a great deal more such usable material in mathematics, and that his ability to express himself was all the poorer for his not being able to draw from this rich source. But it was all, so he thought, quite hopeless, as he was convinced of one thing: he could never penetrate right into the heart of mathematics.

I have often remembered this conversation; it has always suggested avenues of thought to me and plans. I saw immediately that there was something to do here, since in mathematics for me the element of atmosphere had always been the main factor, and this was surely a common source from which the writer and the artist could both draw. I remember an example from my schooldays: some fellow students and I were reading one of Shaw’s plays. We reached the point where the hero asked the heroine what was her secret by means of which she was able to win over and lead the most unmanageable people. The heroine thought for a moment and then suggested that perhaps it could be explained by the fact that she really kept her distance from everyone. At this point the student who was reading the part suddenly exclaimed: ‘That is just the same as the mathematical theorem we learnt today!’ The mathematical question had been: Is it possible to approach a set of points from an external point in such a way that every point of the set is approached simultaneously? The answer is yes, provided that the external point is far enough away from the whole set:



From here you cannot; while you approach some of the points you go farther away from others.

From here you can.

I did not wish to believe the writer’s other statement, namely that he could never penetrate right into the heart of mathematics, that for instance he would never be able to understand the notion of th differential coefficient. I tried to analyse the introduction of this notion into the simplest possible, obvious steps. The result was very surprising; the mathematician cannot even imagine what difficulties the simplest formula can present to the layman. Just as the teacher cannot understand how it is ~~possible that a child can spell c–a–t twenty times, and still not see that it is really a *cat*; and~~ there is more to this than to a cat!

This again was an experience that caused me to do a great deal of thinking. I had always believed that the reason why the public was so ill-informed about mathematics was simply that nobody had written a good popular book for the general public about, say, the differential calculus. The interest patently exists, as the public snaps up everything of this kind that is available to it; but no professional mathematician has so far written such a book. I am thinking of the real professional who knows exactly to what extent things can be simplified without falsifying them, who knows that it is not a question of serving up the usual bitter pill in a pleasanter dish (since mathematics for most is a bitter memory); one who can clarify the essential points so that they hit the eye, and who himself knows the joy of mathematical creation and writes with such a swing that he carries the reader along with him. I am now beginning to believe that for a lot of people even the really popular book is going to remain inaccessible.

Perhaps it is the decisive characteristic of the mathematician that he accepts the bitterness inherent in the path he is travelling. ‘There is no royal road to mathematics’, Euclid said to an interested potentate; it cannot be made comfortable even for kings. You cannot read mathematics superficially; the inescapable abstraction always has an element of self-torture in it, and the one to whom this self-torture is joy is the mathematician. Even the simplest popular book can be followed only by those who undertake this task to a certain extent, by those who undertake to examine painstakingly the details inherent in a formula until it becomes clear to them.

I am not going to write for these people. I am going to write mathematics without formulae. I want to pass on something of the feel of mathematics. I do not know if such an undertaking can succeed. By giving up the formula, I give up an essential mathematical tool. The writer and the mathematician alike realize that form is essential. Try to imagine how you could express the feel of a sonnet without the form of the sonnet. But I still intend to try. It is possible that, even so, some of the spirit of real mathematics can be saved.

One way of making things easy I cannot allow; the reader must not omit, leave for later reading, or superficially skim through, any of the chapters. Mathematics can be built up only brick by brick; here not one step is unnecessary, for each successive part is built on the previous one, even if this is not quite as obviously so as in a boringly systematic book. The few instructions must be carried out, the figures must really be studied, simple drawings or calculations must really be attempted when the reader is asked to do so. On the other hand I can promise the reader that he will not be bored.

I shall not make use of any of the usual school mathematics. I shall begin with counting and I shall reach the most recent branch of mathematics, mathematical logic.

PART I

THE SORCERER’S APPRENTICE

I. Playing with fingers

LET us begin at the beginning. I am not writing a history of mathematics; this could be done only on the basis of written evidence, and how far from the beginning is the first written evidence! We must imagine primitive man in his primitive surroundings, as he begins to count. In these imaginings, the little primitive man, who grows into an educated human being before our eyes, will always come to our aid; the little baby, who is getting to know his own body and the world, is playing with his tiny fingers. It is possible that the words ‘one’, ‘two’, ‘three’ and ‘four’ are mere abbreviations for ‘this little piggie went to market’, ‘this little piggie stayed at home’, ‘this one had roast beef’, ‘this one ha none’ and so on; and this is not even meant to be a joke: I heard from a medical man that there are people suffering from certain brain injuries who cannot tell one finger from another, and with such a injury the ability to count invariably disappears. This connexion, although unconscious, is therefore still extremely close even in educated persons. I am inclined to believe that one of the origins of mathematics is man’s playful nature, and for this reason mathematics is not only a Science, but to at least the same extent also an Art.

We imagine that counting was already a purposeful activity in the beginning. Perhaps primitive man wanted to keep track of his property by counting how many skins he had. But it is also conceivable that counting was some kind of magic rite, since even today compulsion-neurotics use counting as a magic prescription by means of which they regulate certain forbidden thoughts; for example, they must count from one to twenty and only then can they think of something else. Howev this may be, whether it concerns animal skins or successive time-intervals, counting always means that we go beyond what is there by one: we can even go beyond our ten fingers and so emerges man’s first magnificent mathematical creation, the infinite sequence of numbers,

$$1, 2, 3, 4, 5, 6,...$$

the sequence of natural numbers. It is infinite, because after any number, however large, you can always count one more. This creation required a highly developed ability for abstraction, since these numbers are mere shadows of reality. For example, 3 here does not mean 3 fingers, 3 apples or 3 heartbeats, etc., but something which is common to all these, something that has been abstracted fron them, namely their number. The very large numbers were not even abstracted from reality, since no one has ever seen a billion apples, nobody has ever counted a billion heartbeats; we imagine these numbers on the analogy of the small numbers which do have a basis of reality: in imagination one could go on and on, counting beyond any so-far known number.

Man is never tired of counting. If nothing else, the joy of repetition carries him along. Poets are well aware of this; the repeated return to the same rhythm, to the same sound pattern. This is a very live business; small children do not get bored with the same game; the fossilized grown-up will soon find it a nuisance to keep on throwing the ball, while the child would go on throwing it again and again.

We go as far as 4? Let us count one more, then one more, then one more! Where have we got to? To 7 the same number that we should have got to if we had straight away counted 3 more. We have discovered addition

$$4 + 1 + 1 + 1 = 4 + 3 = 7$$

Now let us play about further with this operation: let us add to 3 another 3, then another 3, then another 3! Here we have added 3's four times, which we can state briefly as: four threes are twelve, o in symbols:

$$3 + 3 + 3 + 3 = 4 \times 3 = 12$$

and this is multiplication.

We may so enjoy this game of repetition that it might seem difficult to stop. We can play with multiplication in the same way: let us multiply 4 by 4 and again by 4, then we shall get

$$4 \times 4 \times 4 = 64$$

This repetition or 'iteration' of multiplication is called raising to a power. We say that 4 is the bas and we indicate by means of a small number written at the top right-hand corner of the 4 the number of 4's that we have to multiply; i.e. the notation is this:

$$4^3 = 4 \times 4 \times 4 = 64$$

As is easily seen, we keep getting larger and larger numbers: 4 x 3 is more than 4 + 3, and 4^3 is a good deal more than 4 x 3. This playful repetition carries us well up amongst the large numbers; even more so, if we iterate raising to a power itself. Let us raise 4 to the power which is the fourth power o four:

$$4^4 = 4 \times 4 \times 4 \times 4 = 64 \times 4 = 256$$

and we have to raise 4 to this power:

$$4^{4^4} = 4^{256} = 4 \times 4 \times 4 \times 4 \ldots$$

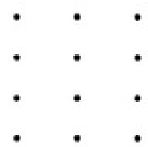
I have no patience to write any more, since I should have to put down 256 4’s, not to mention the actual carrying out of the multiplication! The result would be an unimaginably large number, so that we use our common sense, and, however amusing it would be to iterate again and again, we do not include the iteration of powers among our accepted operations.

Perhaps the truth of the matter is this: the human spirit is willing to play any kind of game that comes to hand, but only those of these mathematical games become permanent features that common sense decides are going to be useful.

Addition, multiplication and raising to powers have proved very useful in man’s common-sense activities and so they have gained permanent civil rights in mathematics. We have determined all those of their properties which make calculations easier; for example, it is a great saving that 7 X 28 can be calculated not only by adding 28 7 times, but also by splitting it into two multiplication processes: 7 X 20 as well as 7 X 8 can quite easily be calculated and then it is readily determined how much 140 + 56 will be. Also in adding long columns of numbers how useful it is to know that no amount of rearranging of the order of the additions is going to spoil the result, as for example 8 + 7 + 2 can be carried out as 8 + 2 = 10 and to 10 it is quite easy to add 7; in this way I have cunningly avoided the awkward addition 8 + 7. We merely have to consider that addition really means counting on by just as much as the numbers to be added and then it becomes clear that changing the order does not alter the result. To be convinced of the same thing about multiplication is a little harder, since 4 x 3 means 3 + 3 + 3 + 3 and 3 x 4 means 4 + 4 + 4 and it is really not obvious that

$$3 + 3 + 3 + 3 = 4 + 4 + 4$$

But this straightaway becomes clear if we do a little drawing. Let us draw four times three dots in these positions . . . one underneath the other



Everyone can see that this is the same thing as if we had drawn three times four dots in the following positions

next to each other. In this way 4 x 3 = 3 x 4. This is why mathematicians have a common name for the multiplier and the multiplicand; the factors.

Let us look at one of the rules for raising to powers:

$$4 \times 4 \times 4 \times 4 \times 4 = 4^5$$

If we get tired of all this multiplying, we can have a little rest; the product of the first three 4’s is 4^3, there is still 4^2 left, so

$$4^3 \times 4^2 = 4^5$$

The exponent of the result is 5, which is 3 + 2; so we can multiply the two powers of 4 by adding thei exponents. As a matter of fact this is always so. For example:

$$5^4 \times 5^2 \times 5^3 = \underbrace{5 \times 5 \times 5 \times 5} \times \underbrace{5 \times 5} \times \underbrace{5 \times 5 \times 5} = 5^9$$

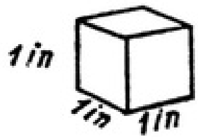
here again 9 = 4 + 2 + 3.

Let us recapitulate the ground we have covered: it was counting that led to the four rules. It could b objected, where does subtraction come in all this? And division? But these are merely reversals of th operations we have had so far (as are extraction of roots and logarithms). Because, for example, 20 ÷ involves our knowing the result of a multiplication sum, namely 20, we are seeking the number whic if multiplied by 5 gives 20 as the result. In this case we succeed in finding such a number, since 5 x 4 = 20. But it is not always easy to find such numbers; in fact it is not even certain that there is one. Fo example 5 does not go into 23 without a remainder since 4 x 5 = 20 is too little and 5 x 5 = 25 is more than 23, and so we are forced to be satisfied with the smaller one and to say that 5 goes 4 times into 23, but 3 is left over. This kind of thing certainly causes more headaches than our playful iterations; the reversed operations are usually bitter operations. It is for this reason that they are favourite points of attack in mathematical research, since mathematicians are well known to take delight in difficulties. So I shall have to return to these reversed or inverse operations in what follows.

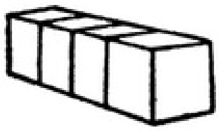
2. *The ‘temperature charts’ of the operations*

WE saw that iteration of operations carried us higher and higher in the realm of large numbers. It is worth spending a little time thinking about just what heights we have reached.

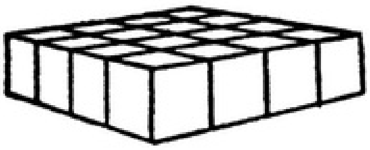
For example, we must raise to a power when we want to calculate the volume of a cube. We choos some small cube as a unit and the question is how many of these small cubes would fill up a bigger cube. Let us take for example an inch cube as our unit, i.e. a cube whose length, breadth and height ar all one inch.



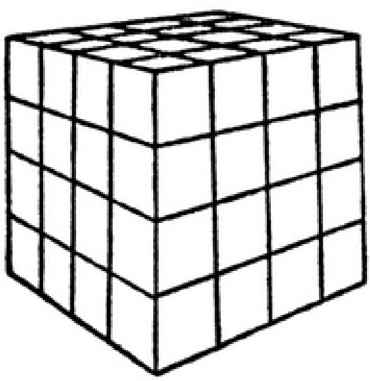
Let us put four of these little cubes next to each other, and we get a row like this:



Then, if we put four such rows next to each other, we make a layer like this:



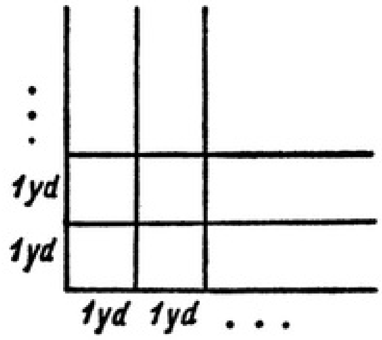
In this there are 4 x 4 = 4^2 cubes. Finally if we put four such layers on top of each other, we shall hav made a big cube like this:



and this is made up of 4 x 4 x 4 = 4^3 = 64 little cubes.

Taking it the other way round, if we start off with the big cube whose length, breadth and height ar 4 inches, this can be made up of 4^3 inch cubes; in general, we get the volume of a cube by raising an edge to the third power. This is why we call raising to the third power cubing.[3]

One consequence of this cubing is that a cube with a relatively short edge will have an enormous volume. For example 1000 yards is not a great distance; everyone can visualize it, if they recall for example that Charing Cross Road in London is about that long. But if we built a cube so that each of its edges was as long as Charing Cross Road, then its volume would be so large that practically the whole of the human race could be accommodated in it. If anyone does not believe this he can make tl following calculations: There are no people taller than say $2\frac{1}{2}$ yards (7' 6"), so at every $2\frac{1}{2}$ yards we should make a floor, and so there would be 400 floors to a height of 1000 yards. If we subdivide these floors lengthwise as well as across into strips a yard wide, like this

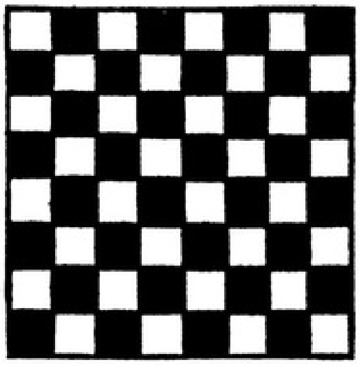


then in every strip we shall have made 1000 squares and there will be 1000 such strips, i.e. there will be 1000 x 1000 = 1,000,000 squares on each floor. The length and the breadth of each square is one yard, we can certainly place five people on each of these squares and so we can squeeze in 1,000,000 times 5, i.e. 5 million people quite well into one of these floors. On the 400 floors there will be 400 times 5 million people, i.e. 2,000,000,000, and this is about as many as there are people in the world, or at least there were not more than this when I was told about this cube before 1943.

And yet in the calculation of the volume of a cube only the third power comes in; a larger exponen carries us much faster still up amongst the large numbers. This fact must have been a great surprise t the potentate from whom the inventor of the game of chess modestly asked for only a few grains of wheat as his reward; he asked for the following to be put on his chessboard of 64 squares, one grain c the first square, twice as many on the second square, i.e. 2; twice as many as that on the third square, i.e. 2 x 2 = 2^2 = 4, and so on. At first this request seems modest enough, but as we run through the squares, we come across higher and higher powers of 2, until finally we are dealing with

$$1 + 2 + 2^2 + 2^3 + 2^4 + \dots + 2^{63}$$

grains of wheat (please imagine that all the powers in between are there as well; I could not be bothered to write in all the 64 terms), and, if someone cares to work out how much this is, he will get so much wheat as a result that the whole surface of the earth could be covered with a half-inch layer c it.



After all this it is not surprising that the iteration of powers carries us up to such enormous heights I shall mention just one fact as a point of interest: it is possible to estimate that 9^{9^9} is such a large number that just for writing it down you would need 11,000 miles of paper (writing five digits in every inch), and a whole lifetime would not be sufficient for its exact calculation.

As I read over what I have written so far, it strikes me that I have been making use of expressions like ‘carries us high up’ amongst the numbers, whereas the number series

1, 2, 3, 4, 5, . . .

is a horizontal series; by right, I should be able to say only that I am going to the right or at the most that I am going forward towards the large numbers. The choice of this particular expression must hav been influenced by the element of atmosphere: to become larger and larger means to grow, and grow gives rise in us to a feeling of breaking through to new heights. The mathematician puts this feeling into concrete form: he often accompanies his imaginings with drawings, and the drawing for very rapid growth is the line that rises steeply upwards.

The sick are very familiar with such drawings; they know that they need only to glance at their temperature charts and this shows the whole progress of their illness. Let us suppose that the following were the temperatures, taken at regular intervals:

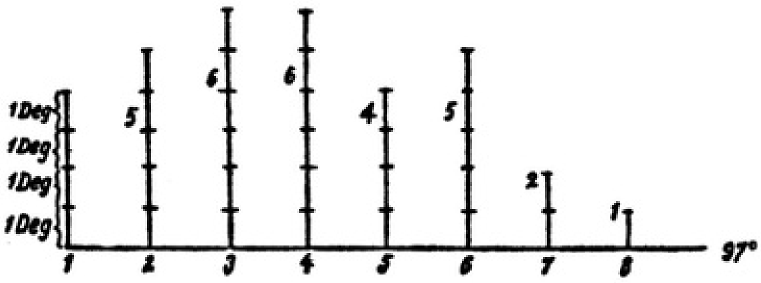
101, 102, 103, 103, 101, 102, 99, 98

These are represented in the following way: first we draw a horizontal line and on this we show the equal time-intervals by equal distances,

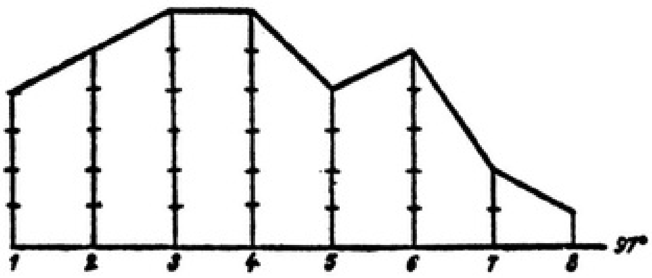
then we choose a certain distance to stand for a degree and from each point of time we draw upwards that multiple of this distance (corresponding to the rising temperature) which is the sick person’s temperature at the point of time in question. But there is no need to draw such long lines, since the temperature never falls below 97, and so we can agree that the height of our horizontal line should correspond to 97. We shall have to draw above this line one after the other

4, 5, 6, 6, 4, 5, 2, 1

degrees. In this way we get the following picture



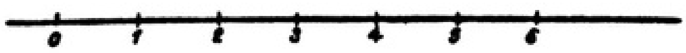
and if we join the end-points of the lines we have drawn, we get



The temperature chart so obtained explains everything. The rising lines indicate the rising of the temperature, the level line shows a stationary period of the illness; the rise in the beginning was steady, this is shown by the fact that the first two joins are equally steep, and so form one straight lin apart from a slight relapse at the sixth time the temperature was taken, the patient improved rapidly: the fall of the line joining the 6th and 7th points is very steep, steeper than any rise.

There is no reason why we should not draw the ‘temperature charts’ of our arithmetical operations

The numbers themselves are usually represented in a similar way along a line: on this line we pick out an arbitrary starting point which we call zero and from this point we measure off equal distances next to one another, i.e. we count in terms of such distances



Anyone who is handy at counting can carry out the operations mechanically on such a line: for example, if we were considering the operation 2 + 3, we need only take 3 steps to the right from the 2 and we can read off the result as being 5. If we were considering 5 – 3 we should take 3 steps to the left from the 5 and so on.

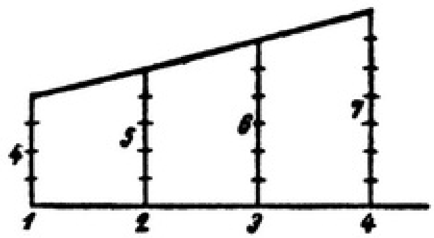
In a similar way is the abacus used, on the wires of which beads can be moved up or down.

But let us leave the horizontal and go upwards. Let us start with a certain number, say 3, and let us see how it grows if we add to it 1, then 2, then 3 and so on, or else if we multiply it by 1, then by 2, then by 3, and finally if we raise it to the first, second and third powers (‘raise’ to a power: in this expression, too, we have the idea of pointing upwards).

Let us begin with addition. One of the terms is always 3, the other variable term will be represente on the horizontal line and the corresponding sum will point upwards

3 + 1 = 4
3 + 2 = 5
3 + 3 = 6
3 + 4 = 7

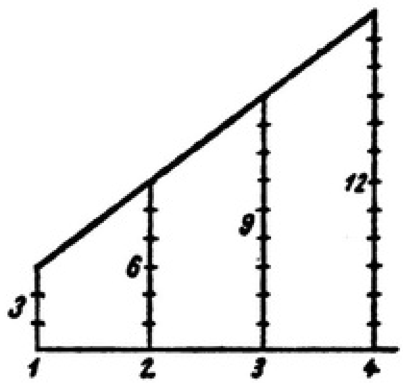
Thus if we represent 1 horizontally by a distance such as this: and vertically by a distance such as this: I the ‘temperature chart’ for addition will be the following:



Here every joining line falls onto one and the same straight line. The sum grows steadily as we increase one of its terms.

In the case of multiplication we have:

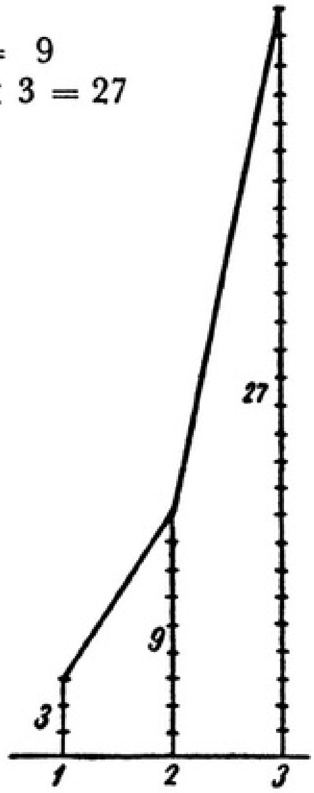
3 x 1 = 3
3 x 2 = 6
3 x 3 = 9
3 x 4 = 12



It can be seen that the product also grows steadily if we increase one of its factors, but much more rapidly than the sum: the straight line we get here is a good deal steeper.

Finally if we take powers we have:

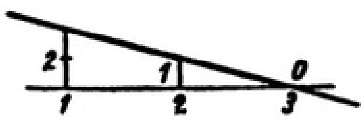
$$3^1 = 3$$
$$3^2 = 3 \times 3 = 9$$
$$3^3 = 3 \times 3 \times 3 = 27$$



The powers do not even grow steadily, but more and more rapidly. There would not even be room for 3^4 on this page. This is the origin of the saying that a certain effect ‘increases exponentially’.

In the same way we can construct the charts for the inverse operations; for example for subtractio we should have:

$$3 - 1 = 2$$
$$3 - 2 = 1$$
$$3 - 3 = 0$$



which gives a falling straight line; so the difference decreases steadily if we increase the term to be subtracted.

Division is rather a delicate operation; I shall return to its. chart at a later stage.

I shall just make one more remark; what we have been engaged in doing here is what the mathematicians call the graphical representation of functions. The sum depends on our choice of valu for the variable term; we express this by saying that the sum is a function of the variable term, and w have represented the growth of this function. In the same way the product is a function of its variable factor, the power of its exponent and so on. Already at the very first operations we have come face to face ~~with functions, and in what follows we shall be examining functional relationships. The no~~tion of function is the backbone of the whole structure of mathematics.

3. *The parcelling out of the infinite number series*

WHAT a long way we have travelled from our games with our fingers! If we have practically forgotten that we have 10 fingers it is only because I did not want to tire my reader with a lot of calculation. Otherwise he would already have noticed that however large a number we write down, w make use of only 10 different symbols, namely,

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

How is it possible to write down any one of the numbers from the infinite number series by using a mere 10 symbols? It is done by parcelling out this indefinitely increasing number series, by enclosin some of its parts: when we have counted 10 units, we say that we can still grasp that amount at a glance. Let us gather them up in one bundle and call such a bundle a ten, the collective name for the 10 units. We can exchange 10 silver shillings for a single 10-shilling note. Now we can count on in longer steps, progressing by tens; then we can bundle together ten tens, for example we could tie a ribbon round them on which we can write ‘1 hundred’. Going on like this, we can bundle 10 hundreds into one thousand, 10 thousands into a tenthousand, 10 tenthousands into a hundred-thousand, and 10 hundredthousands into a million. In this way every number can really be written down with the aid o the above-mentioned ten symbols. When we get beyond 9, we write a 1 again, indicating 1 ten. The number after this consists of one ten and one unit, i.e. it can be written down with the aid of two 1’s. On the other hand, while writing them down we also have to use the words ‘tens’, ‘hundreds’ and similar words. A clever idea makes even these unnecessary: the shopkeeper puts his shilling, two-shilling, half-crown pieces into different sections of his till, the small change on the right because he needs to deal with that a lot in giving change, and towards the left he will put larger and larger denominations. The shopkeeper’s hands get so used to this arrangement that he will know without looking what kind of coin he is picking up, for example, from the third section. In the same way we could agree about the places in which to put the ones, tens, hundreds. Let us write the ones on the right, and then the larger and larger units, moving along towards the left: the second place is for the tens, the third one for the hundreds. In this way we can leave out the words, because we can recogniz the values of the number symbols from their positions; the symbols have thus place-value.

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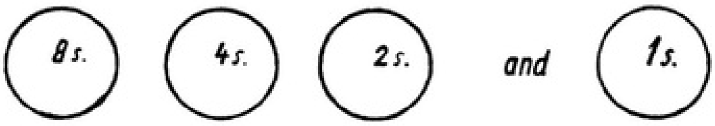
consists of 3 hundreds, 5 tens and 4 ones. This is what we mean when we say that we use the decimal system.

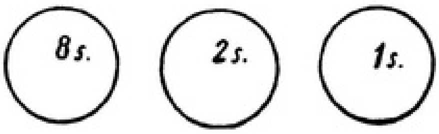
On the other hand there would have been no reason why we should not have stopped before or after 10. I have heard of primitive tribes whose knowledge of counting consists of 1, 2, many. We could build a number system even for them: let us bundle the numbers together in twos. The 2 therefore is a new unit, a two, 2 twos again another unit, a four, 2 fours are eight, and so on. In this number system the symbols

0, 1

are sufficient for writing down any number. We can see this most easily in this way: let us suppose that we have coins like this



in other words the units of the binary number system figure as coins. How could we make up 11*s*. with the smallest possible number of coins? Clearly out of the following three

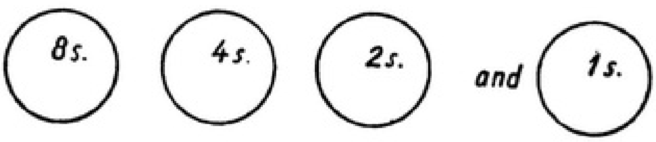


(1s)

which give 11*s*., and out of fewer coins you could never make up 11*s*. Similarly

(2)

make 9*s*., and



(3)