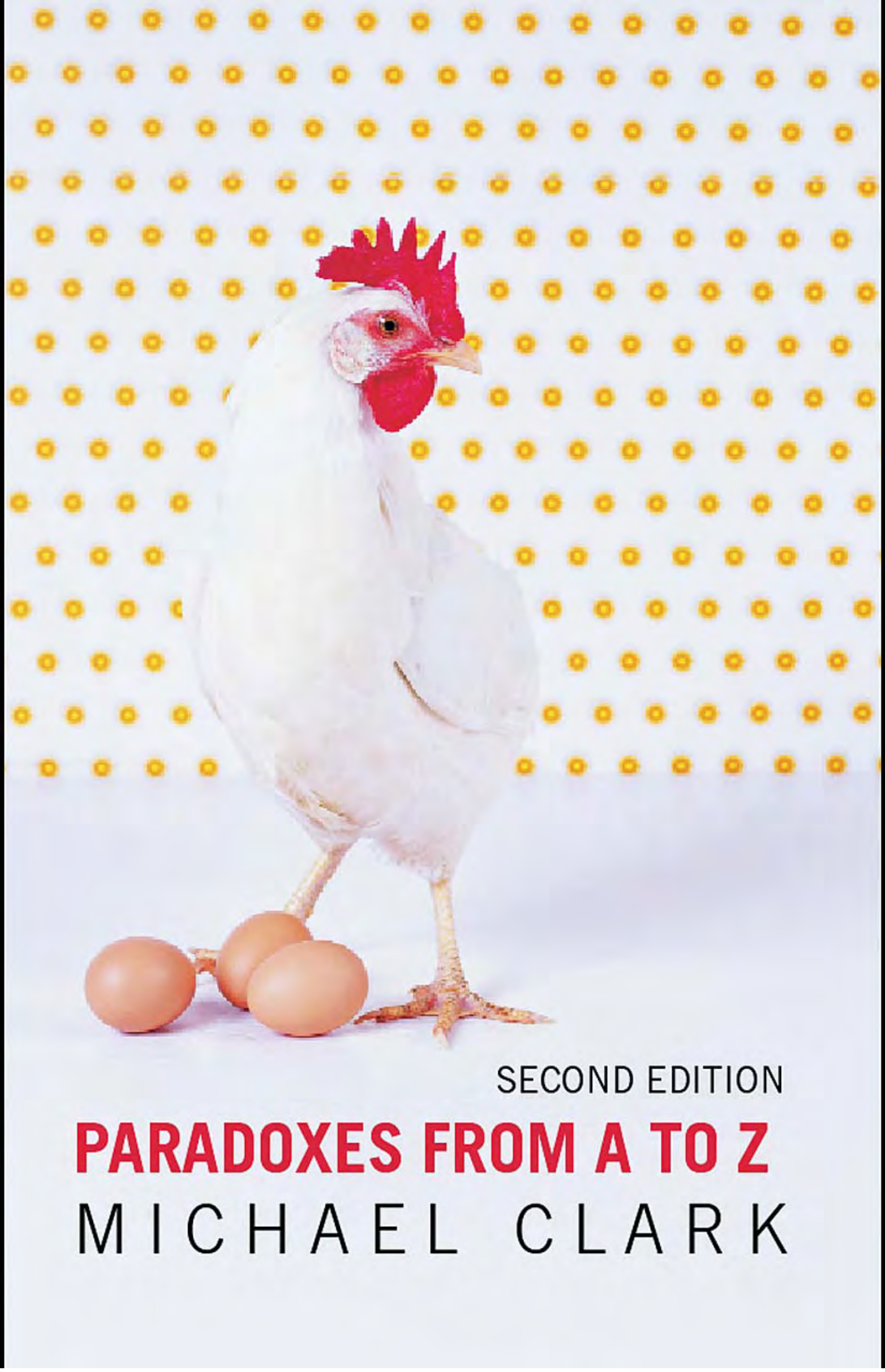
SECOND EDITION
PARADOXES FROM A TO Z
MICHAEL CLARK

Paradoxes from A to Z

Michael Clark's bestselling *Paradoxes from A to Z* is a lively and refreshing introduction to some of the famous puzzles that have troubled thinkers from Zeno and Galileo to Lewis Carroll and Bertrand Russell. He invites you to ponder Achilles and the Tortoise, The Ship of Theseus, Hempel's Ravens, the Prisoners' Dilemma, The Barber Paradox, and many more.

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Alan Weir, *Queen's University Belfast*

Michael Clark is Emeritus Professor of Philosophy at the University of Nottingham, and editor of the journal *Analysis*.

Paradoxes from A to Z

Second edition

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Contents

Foreword to the Second Edition

Ten new paradoxes have been added: **Allais' Paradox**, **the Cable Guy**, **the Charitable Trust**, **the Chicken and the Egg**, **the Paradox of Interesting Numbers**, **the Muddy Children**, **the Numbered Balls**, the recent and striking **Parrondo Paradox**, **the Self-Amendment Paradox** and the **Paradox of Voting**. So there are now entries on 84 paradoxes, as well as the entry on **Paradox** itself. An index has been added and reading lists updated.

The opportunity has been taken to make some corrections and revisions, the most significant being the revision of the paragraph on the third case of Bertrand's (Chord) Paradox. I should like to thank Peter Cave, Bob Kirk, David Miller, Naomi Rosenberg, Nicholas Shackel, Bernard Walliser, Nigel Warburton and the anonymous readers for their comments.

Preface

Pick up a recent issue of a philosophical journal like *Mind* or *Analysis* and it is surprising how many of the papers you see there are about philosophical paradoxes. Philosophy thrives on them, and many have borne abundant fruit. As Quine says, 'More than once in history the discovery of paradox has been the occasion for major reconstruction at the foundation of thought'. The development of nineteenth-century mathematical analysis (Zeno's paradoxes), twentieth-century set theory (the paradoxes of set theory), the limitative theorems of Gödel, and Tarski's theory of truth (the liar group) are dramatic illustrations.

The term *paradox* is given a very broad interpretation in this book, far broader than will appeal to many logical purists. Any puzzle which has been called a 'paradox', even if on examination it turns out not to be genuinely paradoxical, has been regarded as eligible for inclusion, though the most fascinating are among those recognized by the purist, those which appear to be perennially controversial. For a brief discussion of the notion see the entry **Paradox**, p. 151.

This A to Z is a personal selection. But with the exception of two small groups, the well-known philosophical paradoxes will be found here, along with others less familiar or else recently propounded. The first missing group contains a few rather technical set-theoretic paradoxes like Burali–Forti and Zermelo–König, and those of Schrödinger and time-travel involving advanced physics. The other group includes paradoxes I regard as trivial, like the paradox of the omnipotent god who cannot make a stone so heavy that he can't lift it, and near duplicates of some already included. Most of those discussed concern motion, infinity, probability, sets, inference, identity, rationality, knowledge and belief, though there are some from the fields of ethics, political theory and aesthetics.

One entry at least is not a paradox at all, but helps to address the question of what a paradox is.

I have sought to avoid technicality, but many of the most fascinating paradoxes involve infinity, and here it is not possible to avoid technicality completely; the same applies to some of the entries on logical inference. I have tried to explain the basic ideas in simple terms, but if they are still not to your taste there are plenty of wholly non-technical entries to move on to. Most entries are self-contained, although there are frequent cross-references. Where one entry depends on another, this is indicated: for example, the entries on Cantor's and Richard's paradoxes presuppose that those on Galileo's paradox and Hilbert's hotel have been read first, and that on plurality presupposes Galileo's, Hilbert's and Cantor's.

For some paradoxes, for example the Zeno paradoxes of **Achilles and the Tortoise**, **The Arrow** and **The Racecourse**, there is now a broad consensus about their resolution. This is also true of the statistical illusions, like **Bertrand's Box**, **Monty Hall**, **The Xenophobic Paradox** and **Simpson's**. But often the most frivolous-seeming of puzzles turn out to have an unexpected depth and fecundity. It is a mark of such paradoxes (which include **The Liar** group and **The Heap**) that not only is their resolution persistently controversial but their significance is also a matter of debate. The suggested resolutions offered for them here should be read with their controversial nature in mind.

There are two cases in which I have had to provide my own name for a paradox: those I call '**The Paradox of Jurisdiction**' and '**The Xenophobic Paradox**'. A happy consequence was that they enabled me to include paradoxes for the letters 'J' and 'X', which might otherwise have lacked an entry.

Cross-references to other paradoxes are indicated in bold. The *Further Reading* lists have deliberately been kept brief: bibliographies will be found in some of the books cited (A. W. Moore, Sainsbury, Salmon, Sorensen, etc.) Items marked with an asterisk are of a more advanced, and often technical, nature.

Acknowledgements

Thanks to Hao Wang, Stephen Yablo and Zeno for giving their names to paradoxes, and providing entries for ‘W’, ‘Y’ and ‘Z’.

I am grateful to Robert Black, Bob Kirk, Jeff Ketland and John Perry for suggestions and comments, and I have a special debt to Peter Cave, Paul Noordhof, Nick Shackel, Nigel Warburton and two anonymous readers.

If only to provide an example for the paradox of **The Preface**, I have to acknowledge my own responsibility for the errors this book will inevitably contain.

I am grateful to a former editor of *Cogito* for permission to reproduce material from my article ‘An introduction to infinity’, 1992. I have also reproduced material from the appendix to my paper ‘Recalcitrant variants of the liar paradox’, *Analysis*, 1999, vol. 59 (of which I retain the copyright). Ashley Langdon kindly gave permission to use his program to generate the graph on p. 156 from his website.

Achilles and the Tortoise

Achilles runs faster than the tortoise and so he gives it a head start: Achilles starts at d_1 and the tortoise at d_2. By the time Achilles has made up the tortoise's head start and got to d_2, the tortoise is at d_3. By the time Achilles has got to d_3 the tortoise has reached d_4. Each time Achilles makes up the new gap the tortoise has gone further ahead. How can Achilles catch the tortoise, since he has infinitely many gaps to traverse?

This is perhaps the most famous of the paradoxes of Zeno of Elea (born *c.* 490 BC).

Summing an Infinite Series

Of course we know that Achilles will catch the tortoise. He completes the infinitely many intervals in a finite time because each successive interval, being smaller than the one before, is crossed more quickly than its predecessor. Suppose Achilles catches the tortoise after running a mile. The infinitely many smaller and smaller intervals he traverses have to add up to a mile. But how can that be?

It wasn't until the nineteenth century that a satisfactory mathematical way of summing the intervals was developed. The solution was to define the sum of an infinite series as the *limit* to which the sequence of its successive partial sums converges. For simplicity, suppose they both proceed at a uniform speed, and that Achilles goes only twice as fast as the tortoise, giving him a half-mile start. (The principle is the same if Achilles' speed is, more realistically, much greater than twice the tortoise's, but the assumption of a dilatory Achilles makes the arithmetic simpler.)

By the time Achilles has made up this head start, the tortoise has gone a further quarter-mile. When he has gone this further quarter-mile the tortoise is a furlong (one-eighth of a mile) ahead, and so on. Then the intervals Achilles traverses, expressed as fractions of a mile, are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, ... The partial sums are

$\frac{1}{2}$ mile
$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ mile
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ mile
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$ mile
and so on.

So the sequence of partial sums will go:

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128}, \frac{255}{256}, \frac{511}{512}, \frac{1023}{1024}, \frac{2047}{2048}, \frac{4095}{4096}, \ldots$$

It goes on for ever, getting closer and closer ('converging') to 1. In this case 1 is the limit, and so the sum, of the series. Achilles gradually closes in on the tortoise until he reaches it.

More precisely, but in more technical terms, take any number ε greater than 0: then there will be some term in the sequence of finite sums, call it S_j, such that every term from S_j onwards is within ε of the limit. For example, suppose ε is $\frac{1}{8}$. Then every term of the sequence from $\frac{15}{16}$ onwards is within $\frac{1}{8}$ of the limit, 1. If ε is $\frac{1}{1000}$, every term from $\frac{1023}{1024}$ is within $\frac{1}{1000}$ of 1. And so on.

Thomson's Lamp

With Achilles and the tortoise the appearance of paradox arose from the seeming impossibility of completing an infinite series of tasks, a 'supertask'. The following example of a lamp, proposed by the late James Thomson, is a vivid illustration. The lamp is switched on and off alternately: the first switching takes place at $\frac{1}{2}$ minute, the second after $\frac{3}{4}$ minute, and so on. Every time it is switched on it is then switched off, and vice versa. The supertask is completed one minute after it is started. Of course this particular supertask can't physically be performed, but is it impossible in principle? At first Thomson thought he could generate a contradiction out of this description by asking whether the lamp was on or off at one minute: it couldn't be off, because whenever it was turned off it was immediately turned on again, nor could it be on, for a similar reason. But the description of the supertask entails nothing about the lamp's state at one minute, since each switching in the unending series occurs before one minute is up.

Perhaps it stretches our notion of *task* to breaking point to suppose that there is no lower limit whatsoever to the time a task can take. But then the use of the term 'task' for each of Achilles' traversals is tendentious. The claim is only that Achilles can be regarded as having traversed infinitely many intervals in catching the tortoise.

Why, then, is Achilles at the limit, 1, after his infinitely many traversals? After all, none of them gets him to 1, since there is no last traversal. The answer is that, if he is anywhere – as he surely is – he must be at 1, since he can neither be short of 1 nor beyond it. He cannot be short of 1 because he would then still have some – indeed, infinitely many – traversals to make, having so far only made finitely many of the traversals. And he cannot be beyond 1, since there is no interval from 1 to any point beyond 1 which is covered by the traversals.

See also **The Arrow, The Paradox of the Gods, The Tristram Shandy, The Spaceship, The Racecourse.**

Further Reading

Paul Benacerraf, 'Tasks, super-tasks, and the modern Eleatics', *Journal of Philosophy*, 1962, vol. 59, reprinted in Wesley C. Salmon, *Zeno's Paradoxes*, Indianapolis, Bobbs-Merrill, 1970. Salmon's anthology contains other illuminating papers.

R. M. Sainsbury, *Paradoxes*, Cambridge, Cambridge University Press, 2nd edn, 1995, chapter 1.

Wesley C. Salmon, *Space, Time and Motion*, Enrico, California and Belmont, California, Dickenson Publishing Co., Inc., 1975, chapter 2.

Allais' Paradox

You are offered a choice between (1) getting £400 and (2) having a 50% chance of winning £1,000. Although the expected utility of the latter, namely £500, is greater than that of the former, you are more likely to choose (1). A 100% chance of £400 is more attractive than a 50% chance of £1,000 to generally reasonable people.

Up to a point, people who are generally reasonable and prudent prefer a certain gain to an uncertain one with greater expected utility (EU). Thus, (1) they prefer a certain £400 to a 0.5 chance of £1,000, the expected utility of which is £500.

(2) A case involving two choices shows again the psychological advantage of certainty:

A Chance of £2m = 1.0	EU = £2m

B 0.1 chance of £10m	EU = £2.78m
0.89 chance of £2m 0.01 chance of £0	

In B, we get the EU by multiplying the probability by the value for each of the three chances and adding the results together. So we get EU = $(0.1 \times 10) + (0.89 \times 2) + (0.01 \times 0)$ million pounds. (See the **Two-Envelope Paradox** for further explanation.) Most people prefer A to B, because they prefer the certainty of £2m to an uncertain outcome with a greater EU.

C 0.11 chance of £2m	EU =	D 0.1 chance of £10m	EU =
0.89 chance of £0	0.22m	0.9 chance of £0	£1m

But most people prefer D to C (the 0.01 extra chance here of £0 in D has no significant effect) – which is not consistent with the first choice.

(**3**) Given the choice between a certain £2m (EU = £2m) versus a 0.98 chance of £10m and a 0.02 chance of £0 (EU = £9.8m), most people prefer the certainty of the first to the second, although the latter has much greater EU.

But they prefer 0.98 of £10m to 0.01 of £2m, because, in the absence of certainty and a very much higher EU for the former, their psychological values follow probability.

Again, up to a point, people will avoid great loss, however great the expected gain.

(**4**) A comfortably-off person will prefer a 0.999999999 chance of £2m and a 0.000000001 chance of ruin to a 0.999999999 chance of ruin and 0.000000001 chance of gain, *however great the gain*. (Cf. the Sure Loss Principle in the **St Petersburg Paradox**.)

Suppose, for example, the gain were so huge that a tiny chance of it still had a very high EU. For example, suppose the expected gain were a billion (10^9) pounds. This is a 0.000000001 (10^{-9}) chance of a quintillion pounds, a quintillion being a million trillion or 10^{18}. He will still prefer the virtual certainty of £2m. Allais asks rhetorically, 'Who could claim that such a man is irrational? He accepts the first because it gives him a practical certainty of gaining [£2m] and refuses the second because it gives him a practical certainty of being ruined' (Allais, 532, my translation. We ignore here any complication arising from the diminishing marginal utility of money.)

Reasonable, prudent people 'begin in effect by fixing a maximum loss which they do not want in any case to exceed, then they choose by comparing their expected mathematical gain with their chance of ruin, but here it is not a case of mathematical expected gain M of monetary values g, but of mathematical expectation μ of psychological values γ . . .' (533).

In short, people who are generally quite rational tend to be attracted to certainty and averse to loss in the face of mathematical utilities to the contrary, and it would be a mistake to follow the mathematics in our economic planning in ignorance of the psychology. Once this is recognized, the paradox dissolves, but it has helped to teach us an important lesson.

These cases are taken from Allais' paper (referenced below). Maurice Allais, a distinguished French economist and physicist born in 1911, became a Nobel Laureate in Economics in 1988.

A further type of example is given by Kahneman and Tversky, who offer the following 2-stage example:

Stage 1: 0.25 chance of moving to 2nd stage.
Stage 2: Choice between
  A. sure win of $30;
  B. 0.8 chance $45, 0.2 chance of nothing.

You must first choose for both stages. For the second stage, most people will choose A.

When presented in the equivalent, 1-stage, form:

A′ 0.25 chance of $30, 0.75 chance of nothing;
B′ 0.2 chance of $45, 0.8 chance of nothing;

most people will choose B′.

The authors explain this in terms of an 'isolation effect': in the first case people simplify by ignoring the first stage, since it is common to both options, and go for the certainty of $30.

Further Reading

*M. Allais, 'Le comportment de l'homme rationnel devant le risque', *Econometrica*, 1953, vol. 21. (Has English summary at the beginning.)

*Daniel Kahneman and Amos Tversky, 'Prospect theory: an analysis of decision under risk.' *Econometrica*, 1979, vol. 47.

The Paradox of Analysis

We can analyse the notion of brother by saying that to be a brother is to be a male sibling. However, if this is correct, then it seems that it is the same statement as 'To be a brother is to be a brother'. Yet this would mean the analysis is trivial. But surely informative analysis is possible?

This paradox is associated with G. E. Moore (1873–1958), but the problem arose in the work of Gottlob Frege (1848–1925), and it can been traced back to the Middle Ages.

Consider

(1) A brother is a male sibling
(2) Lines have the same direction if and only if they are parallel to one another.

If these analyses are correct then 'brother' and 'male sibling' are synonymous, and so are the expressions for the analysed and analysing notions in the second example. Yet to say (1) is not the same as to say that a brother is a brother – any more than saying (2) is saying that lines have the same direction when they have the same direction.

The paradox poses a threat to the possibility of giving an analysis of a concept, the threat that such an analysis must be either trivial or wrong.

But are 'brother' and 'male sibling' precisely synonymous? As Moore points out, it would be correct to translate the French word *frère* as 'brother' but not as 'male sibling'. And it seems that someone could believe that Pat is a brother without believing he was a male sibling, or believe that two lines had the same direction without believing they were parallel to one another. What then makes an analysis correct? Can examples like (1) and (2) be correct