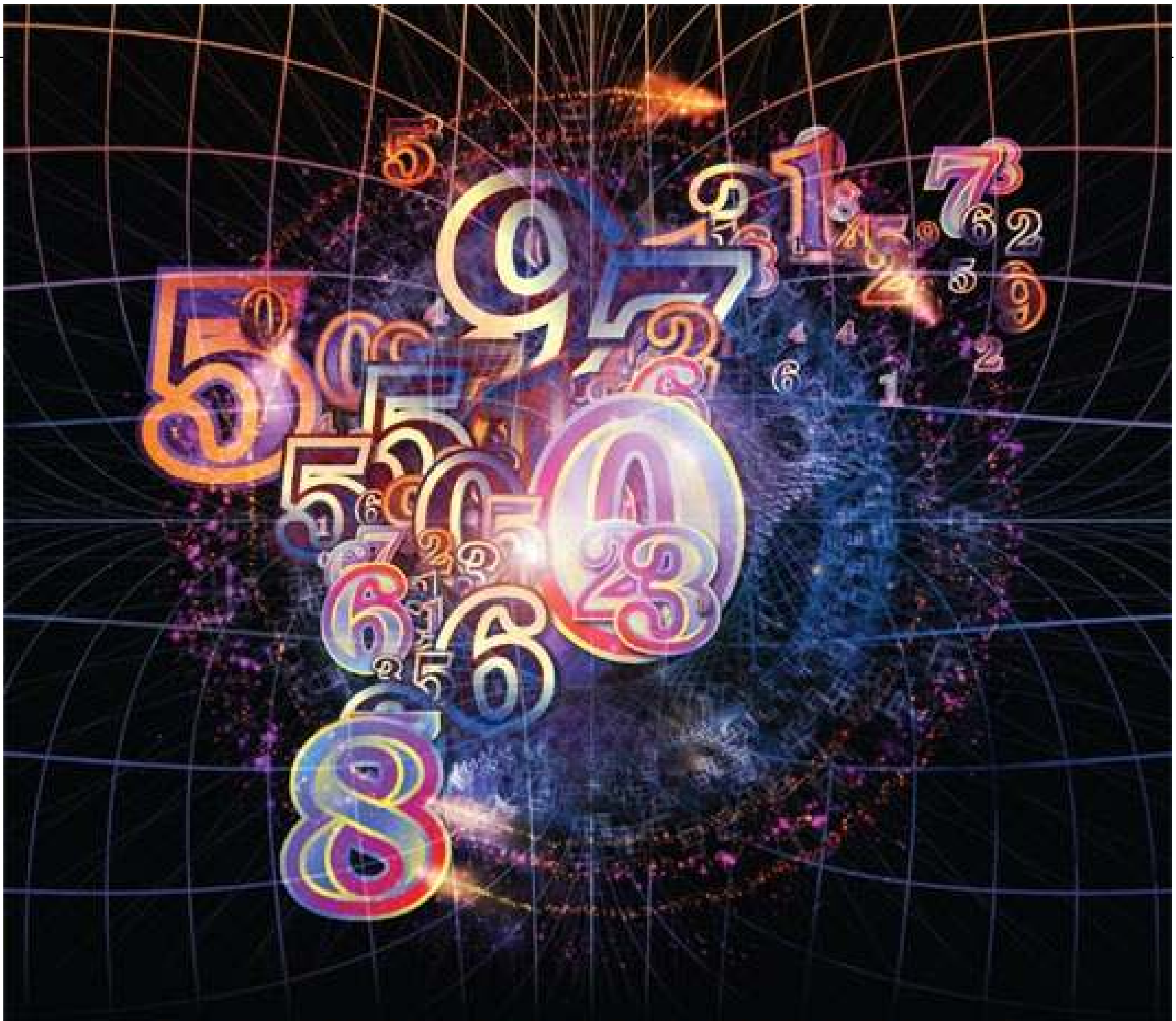

# MATHEMATICAL CURIOSITIES

## A Treasure Trove of Unexpected Entertainments

ALFRED S. POSAMENTIER
INGMAR LEHMANN

## ALSO BY ALFRED S. POSAMENTIER AND INGMAR LEHMANN

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*We dedicate this book of mathematical entertainments to our future generations so that they will be among the multitude that we hope will learn to love mathematics for its power and beauty!*

*To our children and grandchildren, whose future is unbounded:*

*Lisa, Daniel, David, Lauren, Max, Samuel, and Jack.*
*—Alfred S. Posamentier*

*Maren, Tristan, Claudia, Simon, and Miriam.*
*—Ingmar Lehmann*

# CONTENTS

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# INTRODUCTION

It is unfortunate that too many people would be hard-pressed to consider anything mathematical entertainment. Yet with this book we hope to convert the uninitiated general readership to a appreciation for mathematics—and from a very unusual point of view: through a wide variety mathematical curiosities. These include, but certainly are not limited to, peculiarities involvi numbers and number relationships, surprising logical thinking, unusual geometric characteristic seemingly difficult (yet easily understood) problems that can be solved surprisingly simply, curio relationships between algebra and geometry, and an uncommon view of common fractions.

In order to allow the reader to genuinely appreciate the power and beauty of mathematics, navigate these unexpected curiosities in a brief and simple fashion. As we navigate through these tru amazing representations of mathematics, we encounter in the first chapter patterns and relationshi among numbers that a reader on first seeing these will think they are contrived, but they are not. It simply that we have dug out these morsels of fantastic relationships that have bypassed most of during our school days. It is unfortunate that teachers don't take the time to search for some of the beauties, since students in their development stages would see mathematics from a far more favorab point of view.

During its centuries of isolation, the Japanese population was fascinated with *Sangaku* puzzle which we will admire in chapter 2 for the geometry they exhibit. It will allow us to see a curious sid of geometry that may have passed us by as we studied geometry in school. We use these puzzles as gateway to look at some other geometric manifestations.

Problem solving, as most people may recall from their school days, was presented in the form either drill questions or carefully categorized topical problems. In the case of drill, rote memorizatio was expected; while in the case of topical problems, a mechanical response was too often encourag by teachers. What was missing were the many mathematical challenges—problems in a genuine sen —that are off the beaten path, that do not necessarily fit a certain category, that can be very easi stated, and that provide the opportunity for some quite surprisingly simple solutions. These problem provided in the third chapter, are intended to fascinate and captivate the uninitiated!

Measures of central tendency have largely been relegated to the study of statistics—as well th might be. However, when seen from a strictly mathematical point of view—algebraic and geometric- they provide a wonderful opportunity for geometrically justifying algebraic results or algebraical justifying geometric results. We do this in chapter 4, largely in the context of comparing the relativ sizes of the four most popular means, or measures of central tendency, namely, the arithmetic mea (the common average), the geometric mean, the harmonic mean, and the root-mean-square.

When fractions are taught in school, they are presented largely in the context of doing the four bas arithmetic operations with them. In our last chapter, we present fractions from a completely differe standpoint. First recognizing that the ancient Egyptians used only unit fractions (i.e., those in whic the numerator is 1) and the fraction $\frac{2}{3}$. We will present unit fractions in a most unusual way as part a harmonic triangle and eventually leading to Farey sequences. The reader should be fascinated witness that fractions can be about more than just representing a quantity and being manipulated wi others.

Every attempt has been made to make these curiosities as reader friendly, attractive, and motivati as possible so as to convince the general readership that mathematics is all around us and can be fu One by-product of this book is to make the reader more quantitatively and logically aware of th world around him or her.

There are numerous examples in this treasure trove that we hope to have presented in a high intelligible fashion so as to rekindle readers' true love for mathematics, and so that those who mig have been a bit skeptical about whether curiosities that can exhibit the power and beauty mathematics will now go forward and serve as ambassadors for the great field of mathematics. One our goals in this book is to convince the general populace that they should enjoy mathematics and n boast of having been weak in the subject during their school years.

# ARITHMETIC CURIOSITIES

The curiosities found in arithmetic and in numbers, in general, are probably boundless. They rang from peculiarities of certain numbers to number relationships stemming from ordinary arithmet processes. What makes these so entertaining are the unexpected results that are sometim inexplicable. In this chapter we will be presenting to you some of these many arithmetic and numer oddities in mathematics. Some are clearly errors that lead to correct results, while others are corre workings of mathematics that lead to wildly unexpected results. In either case, we hope that throug the mathematics alone you will be entertained without having to apply it to other fields in either th sciences or the real world. Our intent here is to demonstrate a special beauty that can mal mathematics fascinating and enjoyable.

## HOWLERS

In the early years of schooling we learned to reduce fractions to make them more manageable. For th there were specific ways to do it correctly. Some wise guy seems to have come up with a shorter wa to reduce some fractions. Is he right?

He was asked to reduce the fraction $\frac{26}{65}$, and did it in the following way:

$$\frac{2\not{6}}{\not{6}5} = \frac{2}{5}$$

That is, he just canceled out the 6's to get the right answer. Is this procedure correct? Can it b extended to other fractions? If this is so, then we were surely treated unfairly by our elementar school teachers, who made us do much more work. Let's look at what was done here and see if it ca be generalized.

In his book *Fallacies in Mathematics*, E. A. Maxwell refers to these cancellations as “howlers”:

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4} \quad \frac{1\not{9}}{\not{9}5} = \frac{1}{5}$$

Perhaps when someone did the fraction reductions this way, and still got the right answer, it coul just make you howl. This simple procedure continues to give us the correct answers: As we look this awkward—yet easy—procedure, we could begin by reducing the following fractions to lowe terms:

$$\frac{16}{64}, \frac{19}{95}, \frac{26}{65}, \frac{49}{98}.$$

After you have reduced to lowest terms each of the fractions in the usual manner, one may ask wh it couldn't have been done in the following way.

$$\frac{1\cancel{6}}{\cancel{6}4}=\frac{1}{4}$$
$$\frac{1\cancel{9}}{\cancel{9}5}=\frac{1}{5}$$
$$\frac{2\cancel{6}}{\cancel{6}5}=\frac{2}{5}$$
$$\frac{4\cancel{9}}{\cancel{9}8}=\frac{4}{8}=\frac{1}{2}$$

At this point you may be somewhat amazed. Your first reaction is probably to ask if this can be dor to any fraction composed of two-digit numbers of this sort. Can you find another fraction (comprise of two-digit numbers) where this type of cancellation will work? You might cite $\frac{55}{55}=\frac{5}{5}=1$ as a illustration of this type of cancellation. This will, clearly, hold true for all two-digit multiples eleven.

For those readers with a good working knowledge of elementary algebra, we can “explain” th awkward occurrence. That is, why are the four fractions above the only ones (comprised of differe two-digit numbers) where this type of cancellation will hold true?

Consider the fraction $\frac{10x+a}{10a+y}$, in which the second digit of the numerator and the first digit of th denominator match.

The above four cancellations were such that when canceling the $a$'s the fraction was equal to $\frac{x}{y}$.

Let us explore this relationship: $\frac{10x+a}{10a+y}=\frac{x}{y}$.

This yields $y(10x+a) = x(10a+y)$,

$10xy + ay = 10ax + xy$,

$9xy + ay = 10ax \rightarrow y\,(9x + a) = 10ax$,

and so $\frac{10ax}{9x+a}$.

At this point we shall inspect this equation. It is necessary that $x$, $y$, and $a$ are integers since they we digits in the numerator and denominator of a fraction. It is now our task to find the values of $a$ and for which $y$ will also be integral. To avoid a lot of algebraic manipulation, you will want to set up chart that will generate values of $y$ from $y=\frac{10ax}{9x+a}$. Remember that $x$, $y$, and $a$ must be single-dig integers. Below is a portion of the table you will be constructing. Notice that the cases where $x = a$ a excluded, since $\frac{x}{a}=1$.

| $x$ \ $a$ | 1 | 2 | 3 | 4 | 5 | 6 | … | 9 |
|---|---|---|---|---|---|---|---|---|
| 1 | | $\frac{20}{11}$ | $\frac{30}{12}$ | $\frac{40}{13}$ | $\frac{50}{14}$ | $\frac{60}{15}=4$ | | $\frac{90}{18}=5$ |
| 2 | $\frac{20}{19}$ | | $\frac{60}{21}$ | $\frac{80}{22}$ | $\frac{100}{23}$ | $\frac{120}{24}=5$ | | |
| 3 | $\frac{30}{28}$ | $\frac{60}{29}$ | | $\frac{120}{31}$ | $\frac{150}{32}$ | $\frac{180}{33}$ | | |
| 4 | | | | | | | | $\frac{360}{45}=8$ |
| ⋮ | | | | | | | | |
| 9 | | | | | | | | |

**Figure 1.1**

This small portion of the chart (figure 1.1) already generated two of the four integral values of $y$; th is, when $x = 1$, $a = 6$, then $y = 4$, and when $x = 2$, $a = 6$, and $y = 5$. These values yield the fractions and $\frac{26}{65}$, respectively. The remaining two integral values of $y$ will be obtained when $x = 1$ and $a =$ yielding $y = 5$, and when $x = 4$ and $a = 9$, yielding $y = 8$. These yield the fractions $\frac{19}{95}$ and respectively. This should convince you that there are only four such fractions composed of two-dig numbers, excluding two-digit multiples of 11.

Let's extend this idea and investigate whether there are fractions composed of numerators ar denominators of more than two digits where this strange type of cancellation holds true. Try this typ of cancellation with $\frac{499}{998}$. You should find that $\frac{499}{998}=\frac{4}{8}=\frac{1}{2}$.

A pattern is now emerging, and you may realize that

$$\frac{49}{98}=\frac{499}{998}=\frac{4999}{9998}=\frac{49999}{99998}=\frac{4}{8}=\frac{1}{2},$$

$$\frac{16}{64}=\frac{166}{664}=\frac{1666}{6664}=\frac{16666}{66664}=\frac{1}{4},$$

$$\frac{19}{95}=\frac{199}{995}=\frac{1999}{9995}=\frac{19999}{99995}=\frac{1}{5}, \text{ and}$$

$$\frac{26}{65}=\frac{266}{665}=\frac{2666}{6665}=\frac{26666}{66665}=\frac{2}{5}.$$

Enthusiastic readers may wish to justify these extensions of the original howlers. Readers who, at th point, have a further desire to seek out additional fractions that permit this strange cancellation shou consider the following fractions. They should verify the legitimacy of this strange cancellation, ar then set out to discover more such fractions.

$$\frac{332}{830} = \frac{32}{80} = \frac{2}{5}$$

$$\frac{385}{880} = \frac{35}{80} = \frac{7}{16}$$

$$\frac{138}{345} = \frac{18}{45} = \frac{2}{5}$$

$$\frac{275}{770} = \frac{25}{70} = \frac{5}{14}$$

$$\frac{163}{326} = \frac{1}{2}$$

Aside from requiring an algebraic solution, which can be used to introduce a number of importa premises in a motivational way, this topic can also provide some recreational activities. Here are son more of these "howlers."

$$\frac{484}{847} = \frac{4}{7} \quad \frac{545}{654} = \frac{5}{6} \quad \frac{424}{742} = \frac{4}{7} \quad \frac{249}{996} = \frac{24}{96} = \frac{1}{4}$$

$$\frac{48484}{84847} = \frac{4}{7} \quad \frac{54545}{65454} = \frac{5}{6} \quad \frac{42424}{74242} = \frac{4}{7}$$

$$\frac{3243}{4324} = \frac{3}{4} \quad \frac{6846}{8648} = \frac{6}{8} = \frac{3}{4}$$

$$\frac{14714}{71468} = \frac{14}{68} = \frac{7}{34} = \frac{878048}{987804} = \frac{8}{9}$$

$$\frac{1428571}{4285713} = \frac{1}{3} \quad \frac{2857142}{8571426} = \frac{2}{6} = \frac{1}{3} \quad \frac{3461538}{4615384} = \frac{3}{4}$$

$$\frac{767123287}{876712328} = \frac{7}{8} \quad \frac{3243243243}{4324324324} = \frac{3}{4}$$

$$\frac{1025641}{4102564} = \frac{1}{4} \quad \frac{3243243}{4324324} = \frac{3}{4} \quad \frac{4571428}{5714285} = \frac{4}{5}$$

$$\frac{4848484}{8484847} = \frac{4}{7} \quad \frac{5952380}{9523808} = \frac{5}{8} \quad \frac{4274514}{6428571} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{5454545}{6545454} = \frac{5}{6} \quad \frac{6923076}{9239768} = \frac{6}{8} = \frac{3}{4} \quad \frac{4242424}{7424242} = \frac{4}{7}$$

$$\frac{5384615}{7538461} = \frac{5}{7} = \quad \frac{2051282}{8205128} = \frac{2}{8} = \frac{1}{4} \quad \frac{3116883}{8311388} = \frac{3}{8}$$

$$\frac{6486486}{8648648} = \frac{6}{8} = \frac{3}{4} \quad \frac{484848484}{848484847} = \frac{4}{7}$$

This peculiarity shows how elementary algebra can be used to investigate an amusing number-theo situation. These are just some of the hidden treasures that mathematics continues to hold and that w will explore as we journey through this chapter.

## A PAINTING TITLED *A DIFFICULT ASSIGNMENT*

At the end of the nineteenth century, the Russian artist Nikolai Petrovich Belsky[1] (1868–194 produced a painting with the title *A Difficult Assignment*. In the painting (figure 1.2), we see a grou of students mulling around a chalkboard, apparently frustrated with an assignment of calculating a arithmetic challenge.

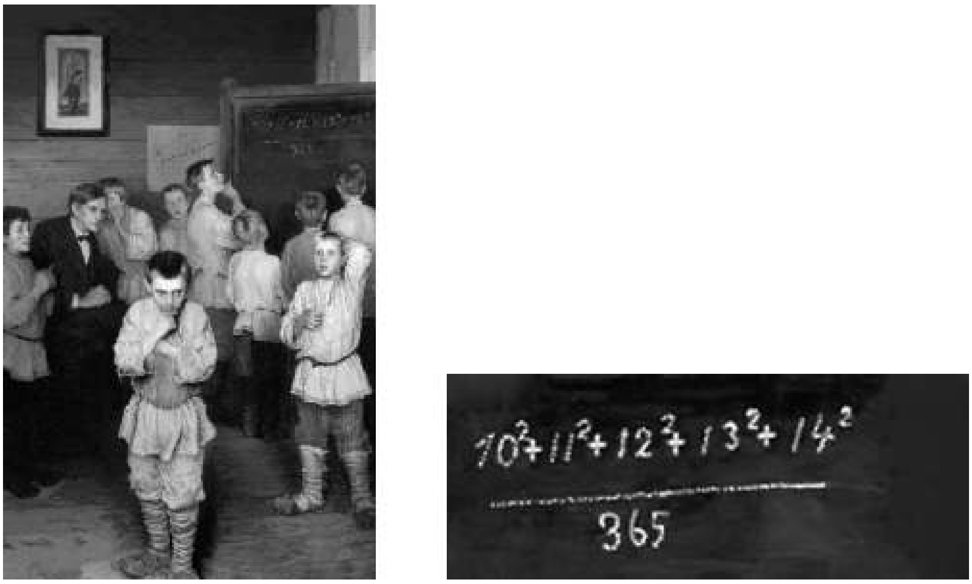


**Figure 1.2a and 1.2b: Nikolai Petrovich Belsky's: *A Difficult Assignment* (1895).**

The problem is to find the value of $\frac{10^2+11^2+12^2+13^2+14^2}{365}$.

Imagine trying to solve this problem without a calculator. It is certainly doable but somewhat tim consuming. However, through the amazing relationships that exist among numbers, we can see th following property that we can exploit. By partitioning the five numbers to be squared, we find th the sum of the first three squares has the same sum as the next two squares. In each case, the sum 365, which then trivializes the original exercise.

$$\frac{(10^2+11^2+12^2)+(13^2+14^2)}{365}=\frac{365+365}{365}=2.$$

Those who recognize this pattern might also be aware of the following pattern:

$$3^2 + 4^2 = 5^2 \quad (= 25),$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2 \quad (= 365),$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2 \quad (= 2{,}030).$$

First, you will notice that on the left side of the equals sign in each case we have one more term tha we have on the right side, and the numbers being squared are consecutive.

An ambitious reader might try to find the next equation, where five squared numbers would be o the left side of the equals sign and four on the right side.

Although it might be easier to do this with a calculator, it may be more fun to look for a pattern t make our calculation even easier.

## THE MAGIC OF ALGEBRA

There are times when an overwhelming arithmetic problem can be nicely simplified with some bas algebra. Let us consider one such example now. In today's world, complicated calculations are easi disposed of using a calculator. However, it is entertaining to see how using algebraic manipulation ca make a very complicated calculation practically trivial.

~~Consider the task of finding the value of $\sqrt{1999 \cdot 2000 \cdot 2001 \cdot 2002 + 1}$. Surely, using a calculator,~~ we ca find that this cumbersome expression is equal to 4,001,999. However, it is interesting to see how v can generalize this expression to our advantage. Since the numbers being multiplied are consecutiv let's see if that gives us an advantage. We begin by letting $n$ = 2,000 and express the other numbe under the radical sign in this way:

$$(n - 1) \cdot n \cdot (n + 1) \cdot (n + 2) + 1.$$

Now for some algebraic gymnastics: by multiplying the terms of this long algebraic expression ar then adding 1, we get:

$$(n - 1) \cdot n \cdot (n + 1) \cdot (n + 2) + 1 = n^4 + 2n^3 - n^2 - 2n + 1.$$

We shall now rearrange and dismantle these terms to suit our plan to get a workable expression:

$n^4 + 2n^3 - n^2 - 2n + 1 = n^4 + n^3 - n^2 + n^3 + n^2 - n - n^2 - n + 1.$

This allows us to form the following product of two trinomials:

$$n^4 + n^3 - n^2 + n^3 + n^2 - n - n^2 - n + 1 = (n^2 + n - 1) \cdot (n^2 + n - 1) = (n^2 + n - 1)^2.$$

Replacing the original term under the radical sign with its equivalent established above, we are ab to simplify the expression under the radical—a perfect square—which allows us to remove the radic sign. $\sqrt{(n-1) \cdot n \cdot (n+1) \cdot (n+2) + 1} = \sqrt{(n^2 + n - 1)^2} = |n^2 + n - 1|$.

Because we are working with natural numbers, we can conclude tha $\sqrt{(n-1) \cdot n \cdot (n+1) \cdot (n+2) + 1} = \sqrt{(n^2 + n - 1)^2} = n^2 + n - 1$.

Therefore, when $n$ = 2,000, we get $\sqrt{1999 \cdot 2000 \cdot 2001 \cdot 2002 + 1} = 2000^2 + 2000 - 1 = 4{,}000{,}000 + 2{,}000 - 1 = 4{,}001{,}999$, which is what we expected, since conforms to the result obtained by using a calculator.

We have seen how algebra can help us understand and also simplify arithmetic processes, and w are now ready to explore some peculiarities embedded with particular numbers. Many numerals v take for granted we view only as the quantities they represent. Here are some rather-curious numer insights and relationships that may make you consider and appreciate these numbers differently.

## THE CURIOUS NUMBER 8

The number 8, which, in the Chinese culture, is the “lucky” number, has a unique arithmetic feature: is the only cube number that is smaller than a square number by 1. That is, $8 = 2^3 = 9 - 1 = 3^2 - 1$.

## THE CURIOUS NUMBER 9

The number 9 is the only square number that is equal to the sum of the cubes of two consecuti natural numbers. That is to say, $9 = 1^3 + 2^3$.

While we are considering the sum of the cubes, we can recall the finding by the famous Swi mathematician Leonhard Euler (1707–1783), who stated that the smallest natural number that can b expressed as the sum of the cubes of natural numbers in two ways is 1,729.

That is, $1^3 + 12^3 = 1 + 1{,}728 = 1{,}729$, and $9^3 + 10^3 = 729 + 1{,}000 = 1{,}729$. Now returning to th number 9, we find that it can be expressed in fractional form using all ten digits exactly once, as v see in the following fractions, which are the only fractions that will give us this amazing resul $9 = \frac{95742}{10638}$, $9 = \frac{95823}{10647}$, and $9 = \frac{97524}{10836}$. However, if we allow the 0 to take a first position in any of th numbers, we get an additional three fractional equivalents to the number 9 also using all the digi exactly once.

$9 = \frac{57429}{06381}$, $9 = \frac{58239}{06471}$, and $9 = \frac{75249}{08361}$.

## THE CURIOUS NUMBER 11

The number 11 is truly a curious number. According to the British king George V, the armistice i 1918 occurred at the eleventh hour of the eleventh day of the eleventh month of the year.

In the American measuring system, the number 11 appears as a factor in linear measurements a follows: 11·20 yards = 1 furlong, and 11·160 yards = 1 mile.

We should also note that the number 11 is the only palindromic prime number that has an eve number of digits. Some more curiosities with the number 11 are offered here to amuse the read further.

First, we have $11^2$ equal to the sum of five consecutive powers of 3 as follows:

$$11^2 = 121 = 3^0 + 3^1 + 3^2 + 3^3 + 3^4.$$

Then we have $11^3$ as the sum of the squares of three consecutive odd numbers:

$$11^3 = 1{,}331 = 19^2 + 21^2 + 23^2.$$

We can also express the number 11 as the sum of a square and a prime in two different ways—and it the smallest number that has this property!

$$11 = 2^2 + 7$$
$$11 = 3^2 + 2$$

Now here is one property of 11 that is really spectacular: if we reverse the digits of any number, whic is divisible by 11, the resulting number will also be divisible by 11. To demonstrate this, let us tak for an example the number 135,916, which is 11 times 12,356, and reverse the digits to get 619,53 which just happens to be 11 times 56,321, clearly a multiple of 11. You may wish to try this with othe multiples of 11 and entertain your friends with this peculiarity.

Here is a nice little trick involving the number 11. Take any number where no two adjacent digi have a sum greater than 9. Then multiply that number by 11, and reverse the digits of this produc Then divide this result by 11. The resulting number will be the reverse of the original number. Let u consider as an example the number 235,412, which is a number where no two adjacent digits have th sum greater than 9. When we multiply it by 11 we get 235,412·11 = 2,589,532. Reversing the digits o this product, we get 2,359,852; and then dividing it by 11, we get 214,532, which is a number whos digits are in the reverse order of the original number.

Aside from being the fifth prime number, the number 11 is also the fifth Lucas number. You ma recall the *Lucas numbers* are a sequence of numbers beginning with 1 and 3, where each succeedi number is the sum of the two previous numbers, as in the following sequence: 1, 3, 4, 7, 11, 18, 29, 4 76, 123,…The sequence was popularized by the French mathematician Edouard Lucas (1842–1891 who also brought popularity to the Fibonacci numbers from which he got the idea of this sequence.[2]

To the left of the Pascal triangle in figure 1.3, you will notice that the sum of the numbers in eac row generates the powers of 2, while the oblique sums show the *Fibonacci numbers*, which are simil to the Lucas numbers in that they are also generated by the sum of consecutive numbers; however, th time beginning with 1 and 1 as the first two numbers. They are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,…

~~We can also find the powers of 11 on the first few rows of the famous Pascal triangle as shown i~~ figure 1.3. Up to the fifth row, the powers of 11 appear directly: $11^4 = 14{,}641 = 1 \cdot 10^4 + 4 \cdot 10^3 + 6$ $10^2 + 4 \cdot 10^1 + 1 \cdot 10^0$.

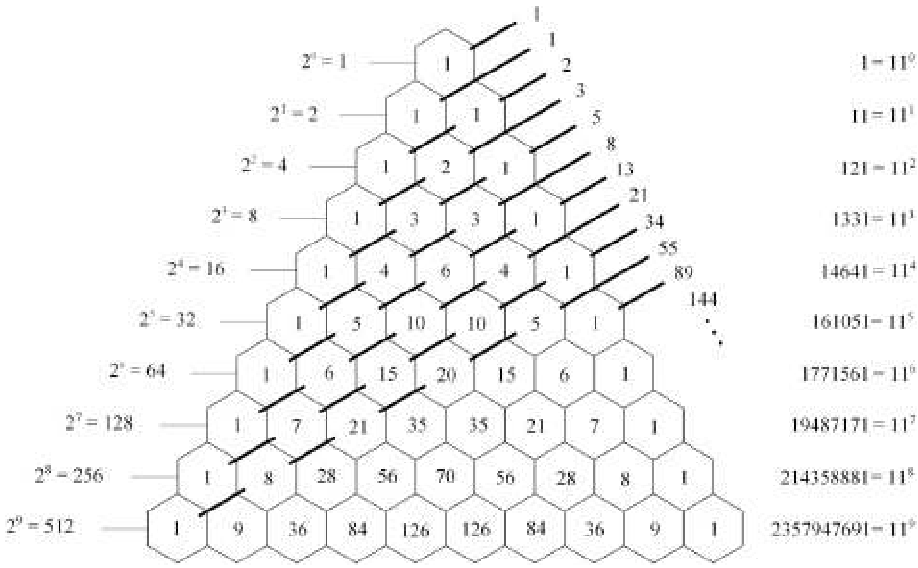


**Figure 1.3**

If we try to get the fifth power of 11, we notice that this row has two-digit numbers, so we will need t carry over the tens digit of each of these two-digit numbers to the next place (to the left) to get $11^5$ 161,051.

| | | | | | |
|---|---|---|---|---|---|
| 1 | **5** | **10** | **10** | **5** | **1** |
| 1 | 5 | 10+1 | 0 | 5 | 1 |
| 1 | 5+1 | 1 | 0 | 5 | 1 |
| **1** | **6** | **1** | **0** | **5** | **1** |

You may get a better understanding of this from the following:

$$\mathbf{1} \cdot 10^5 + \mathbf{5} \cdot 10^4 + \mathbf{10} \cdot 10^3 + \mathbf{10} \cdot 10^2 + \mathbf{5} \cdot 10^1 + \mathbf{1} \cdot 10^0$$
$$= 1 \cdot 10^5 + 5 \cdot 10^4 + 10 \cdot 10^3 + 1 \cdot 10^3 + 0 \cdot 10^2 + 5 \cdot 10^1 + 1 \cdot 10^0$$
$$= 1 \cdot 10^5 + (5 + 1) \cdot 10^4 + 1 \cdot 10^3 + 0 \cdot 10^2 + 5 \cdot 10^1 + 1 \cdot 10^0$$
$$\mathbf{= 1} \cdot 10^5 + \mathbf{6} \cdot 10^4 + \mathbf{1} \cdot 10^3 + \mathbf{0} \cdot 10^2 + \mathbf{5} \cdot 10^1 + \mathbf{1} \cdot 10^0$$
$$= 161{,}051 = 11^5.$$

Analogously, we get $11^6 = 1{,}771{,}561$ in the same way:

| **1** | **6** | **15** | **20** | **15** | **6** | **1** |
|---|---|---|---|---|---|---|
| 1 | 6 | 15 | 20 + 1 | 5 | 6 | 1 |
| 1 | 6 | 15 + 2 | 1 | 5 | 6 | 1 |
| 1 | 6 + 1 | 7 | 1 | 5 | 6 | 1 |
| **1** | 7 | 7 | **1** | **5** | **6** | **1** |

Let's discuss a very nifty way to multiply by 11. This technique always enchants the unsuspectin mathematics-phobic person, because it is so simple that it is even easier than doing it on a calculator

The rule is very simple: *To multiply a two-digit number by 11, just add the two digits and place th sum between the two digits*.

Let's try using this technique. Suppose you wish to multiply 45 by 11. According to the rule, add and 5 and place the sum between the 4 and 5 to get 495.

This does get a bit more difficult when the sum of the two digits you added results in a two-dig number. What do we do in a case like that? We no longer have a single digit to place between the tw original digits. So if the sum of the two digits is greater than 9, we place the units digit between t two digits of the number being multiplied by 11 and “carry” the tens digit to be added to the hundre digit of the product. Let's try it with 78 · 11. We begin by taking 7 + 8 = 15. We place the 5 betwee the 7 and 8, and add the 1 to the 7, to get [7 + 1][5][8] or 858.

You may legitimately ask if the rule also holds when 11 is multiplied by a number of more than tw digits. Let's try for a larger number such as 12,345 and multiply it by 11 to see if our method sti works.

Here we begin at the right-most digit and add every pair of digits, moving to the left.

1[1 + 2][2 + 3][3 + 4][4 + 5]5 = 135,795.

Recall what results when we reverse the digits of this multiple of 11 to get 597,531. When we divi this number by 11, we get 54,321. Notice how this is the reverse of the multiplier above, which w 12,345. An ambitious reader may want to determine when the multiplier will be the reverse as is th case here.

Returning now to our nifty technique for multiplying by 11, we consider a number where the sum o two consecutive digits is greater than 9. Here we use the procedure described before: place the uni digit appropriately and carry the tens digit. We will do one of these here.

Multiply 456,789 by 11.

We carry the process step by step:

4[4+5][5+6][6+7][7+8][8+9]9
4[4+5][5+6][6+7][7+8][17]9
4[4+5][5+6][6+7][7+8+*1*][7]9
4[4+5][5+6][6+7][16][7]9
4[4+5][5+6][6+7+*1*][6][7]9
4[4+5][5+6][14][6][7]9
4[4+5][5+6+*1*][4][6][7]9
4[4+5][12][4][6][7]9

4[4+5+*1*][2][4][6][7]9
4[10][2][4][6][7]9
[4+*1*][0][2][4][6][7]9
[5][0][2][4][6][7]9
5,024,679

This rule for multiplying by 11 ought to be shared with your friends. Not only will they be impresse with your cleverness, they may also appreciate knowing this shortcut—and, above all, it will mal you a good ambassador for mathematics.

At the oddest times the issue can come up to determine if a given number is divisible by 11. If yo have a calculator at hand, the problem is easily solved. But that is not always the case. Besides, the is such a clever "rule" for testing for divisibility by 11 that it is worth knowing just for its charm, n to mention its utility.

The rule is quite simple: *If and only if the difference of the sums of the alternate digits is divisib by 11, then the original number is also divisible by 11*.

This may sound a bit complicated, but it really isn't. Let us take this rule one piece at a time. Th sums of the alternate digits means you begin at one end of the number, taking the first, third, and fif digits (and so on) and add them. Then, for the second sum, add the remaining (even-placed) digit Subtract the two sums, and inspect the difference for divisibility by 11.

This may be best demonstrated through an example. We shall test 768,614 for divisibility by 1 Sums of the alternate digits are: 7 + 8 + 1 = 16, and 6 + 6 + 4 = 16. The difference of these two sum 16 – 16 = 0, which is divisible by 11. Therefore, we can conclude that 768,614 is divisible by 11.

Another example might be helpful to firm up your understanding of this procedure. To determine 918,082 is divisible by 11, we find the sums of the alternate digits: 9 + 8 + 8 = 25, and 1 + 0 + 2 = Their difference is 25 – 3 = 22, which is divisible by 11, and so again we can conclude that the numb 918,082 is divisible by 11.[3]

## NUMBERS WHERE ALL THE DIGITS ARE ONES

Having seen some of the unusual features of the number 11, let's consider large numbers consisting only 1s—called *repunits*.[4]

The next larger number after 11 that consists of all 1s is the number 111, and it, too, has som curious properties.

It is the third difference of two squares, and the number 1,111 is the fourth difference of tw squares. We find that the progression of such differences of squares leads to repunit numbers:

$1^2 - 0^2 = 1$
$6^2 - 5^2 = 11$
$20^2 - 17^2 = 111$
$56^2 - 45^2 = 1,111$
$156^2 - 115^2 = 11,111$
$556^2 - 445^2 = 111,111$
$344^2 - 85^2 = 111,111$
$356^2 - 125^2 = 111,111$

Within this list of numbers that will result in 1s, we see a pattern emerging. Look at the secon fourth, and sixth entries. You will notice an additional pattern between the generating numbers. Eac time an additional 5 and 4 is tagged onto the front of the numbers, respectively, we create anoth repunit. If we continue this pattern, notice what a spectacular pattern evolves.

$$6^2 - 5^2 = 11$$
$$56^2 - 45^2 = 1{,}111$$
$$556^2 - 445^2 = 111{,}111$$
$$5556^2 - 4445^2 = 11{,}111{,}111$$
$$55556^2 - 44445^2 = 1{,}111{,}111{,}111$$
$$555556^2 - 444445^2 = 111{,}111{,}111{,}111$$
$$5555556^2 - 4444445^2 = 11{,}111{,}111{,}111{,}111$$
$$55555556^2 - 44444445^2 = 1{,}111{,}111{,}111{,}111{,}111$$
$$555555556^2 - 444444445^2 = 111{,}111{,}111{,}111{,}111{,}111$$

$$\ldots$$
$$5555555555555556^2 - 4444444444444445^2$$
$$= 1{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111$$

Of this list, the only prime number is 11. In fact, the next two prime numbers of all 1s a 1,111,111,111,111,111,111, and 11,111,111,111,111,111,111,111. It is quite obvious that these last two numbers will be prime in any arrangement of the digits—since they are all the same!

However, we should be aware that there are, in fact, prime numbers, where all arrangements of the digits result in another prime number. The first few of these are 11, 13, 17, 37, 79, 113, 199, and 33 You might like to find the next few such primes that create new prime numbers with each arrangeme of their digits.

The story about our "curious number 11" continues as we examine the numbers generated by th difference of squares that are multiples of numbers consisting of numbers containing only 1s.

$$7^2 - 4^2 = 33 = 3 \cdot 11$$
$$67^2 - 34^2 = 3{,}333 = 3 \cdot 1{,}111$$
$$667^2 - 334^2 = 333{,}333 = 3 \cdot 111{,}111$$
$$6667^2 - 3334^2 = 33{,}333{,}333 = 3 \cdot 11{,}111{,}111$$
$$66667^2 - 33334^2 = 3{,}333{,}333{,}333 = 3 \cdot 1{,}111{,}111{,}111$$

Here is another such pattern of numbers that should be admired.

$$8^2 - 3^2 = 55 = 5 \cdot 11$$
$$78^2 - 23^2 = 5555 = 5 \cdot 1{,}111$$
$$778^2 - 223^2 = 555{,}555 = 5 \cdot 111{,}111$$
$$7778^2 - 2223^2 = 55{,}555{,}555 = 5 \cdot 11{,}111{,}111$$
$$77778^2 - 22223^2 = 5{,}555{,}555{,}555 = 5 \cdot 1{,}111{,}111{,}111$$

On further investigation of repunit numbers, we discover an interesting pattern, one in which w divide 111,111,111 by 9 to give us the number 12,345,679. Notice we have the digits in numerica order, but we are missing the digit 8. Yet when we consider the following pattern, the 8 is once agai included in generating numbers consisting of only 1s.

$$\begin{array}{rcccccl} 0 & \cdot & 9 & + & 1 & = & 1 \\ 1 & \cdot & 9 & + & 2 & = & 11 \\ 12 & \cdot & 9 & + & 3 & = & 111 \\ 123 & \cdot & 9 & + & 4 & = & 1{,}111 \\ 1{,}234 & \cdot & 9 & + & 5 & = & 11{,}111 \\ 12{,}345 & \cdot & 9 & + & 6 & = & 111{,}111 \\ 123{,}456 & \cdot & 9 & + & 7 & = & 1{,}111{,}111 \\ 1{,}234{,}567 & \cdot & 9 & + & 8 & = & 11{,}111{,}111 \\ 12{,}345{,}678 & \cdot & 9 & + & 9 & = & 111{,}111{,}111 \end{array}$$

Don't stop now—continue this pattern to the following:
123,456,789 · 9 + 10 = 1,111,111,111

As you can see, repunit numbers (sometimes also referred to as unit-digit numbers) seem to genera some rather-interesting relationships and patterns. Let us investigate what happens when we take th square of successive unit-digit numbers as shown in figure 1.4.

| Number of 1's | $n$ | $n^2$ |
|---|---|---|
| | | |
| 1 | 1 | 1 |
| 2 | 11 | 121 |
| 3 | 111 | 12321 |
| 4 | 1111 | 1234321 |
| 5 | 11111 | 123454321 |
| 6 | 111111 | 1234564321 |
| 7 | 1111111 | 1234567654321 |
| 8 | 11111111 | 123456787654321 |
| 9 | 111111111 | 12345678987654321 |
| 10 | 1111111111 | 1234567900987654321 |

**Figure 1.4**

To get a better view of these repunit numbers, $r_n$, we will factor them into their prime factors follows:

| | | | | | |
|---|---|---|---|---|---|
| $r_1$ = | 1 = | 1 | $r_{11}$ = | 11111111111 = | 21,649·513,239 |
| $r_2$ = | 11 = | 11 | $r_{12}$= | 111111111111 = | 3·7·11·13·37·101·9,901 |
| $r_3$ = | 111 = | 3·37 | $r_{13}$= | 1111111111111 = | 53·79·265,371,653 |
| $r_4$ = | 1111 = | 11·101 | $r_{14}$= | 11111111111111 = | 11·239·4,649·909,091 |
| $r_5$ = | 11111 = | 41·271 | $r_{15}$ = | 111111111111111 = | 3·31·37·41·271·2,906,161 |
| $r_6$ = | 111111 = | 3·7·11·13·37 | $r_{16}$ = | 1111111111111111 = | 11·17·73·101·137·5,882,353 |
| $r_7$ = | 1111111 = | 239·4,649 | $r_{17}$ = | 11111111111111111 = | 2,071,723·5,363,222,357 |
| $r_8$= | 11111111 = | 11·73·101·137 | $r_{18}$= | 111111111111111111 = | $3^2$·7·11·13·19·37·52,579·333,667 |
| $r_9$ = | 111111111 = | $3^2$ ·37·333,667 | $r_{19}$ = | 1111111111111111111 = | 1111111111111111111 |
| $r_{10}$= | 1111111111 = | 11·41·271·9,091 | $r_{20}$ = | 11111111111111111111 = | 11·41·101·271·3,541·9,091·27,961 |

We notice that $r_2$ and $r_{19}$ are prime numbers. The question then arises, are there other such repuni that are primes? The answer is yes. However, mathematicians have struggled with this question fo years. For example, the German mathematician Gustav Jacob Jacobi (1804 –1851) pursued th question as to whether the repunit $r_{11}$ is a prime number. Today, a computer algebra system ca answer this question in less than a second. Factoring repunit numbers is often very difficult; howeve with the aid of a computer we find that the repunit $r_{71}$ is factorable as follows and, therefore, is not prime number.

$r_{71}$ = 11111111111111111111111111111111111111111111111111111111111111111111111
= 241573142393627673576957439049 · 45994811347886846310221728895223034301839.

By 1930, it was known that $r_2$ and $r_{19}$ (Oscar Hoppe, 1916) as well as $r_{23}$ (Lehmer and Kraitchi 1929) are prime numbers.[5] In 1970, a mathematics student, E. Seah, was able to demonstrate that th repunit $r_{317}$ is also a prime number. The search for a prime number among the repunits continues. Fo example, in 1985 the repunit $r_{1,031}$ was discovered to be a prime by H. C. Williams and H. Dubne Further repunits that have been identified as primes are: $r_{49,081}$ (H. Dubner, 1999), $r_{86,453}$ (L. Baxte 2000), $r_{109,297}$ (P. Bourdelais and H. Dubner, 2007), and $r_{270,343}$ (M. Voznyy and A. Budnyy, 2007). It is believed today that there are an infinite number of repunits that are prime numbers.

## THE SPECIAL NUMBER TRIO: 16, 17, 18

The three numbers 16, 17, and 18 have a curious relationship. First of all, let us look at the pair numbers 16 and 18, which have a very special relationship. Each of these numbers can represent th area of a rectangle that is numerically equal to its perimeter. That is to say that a rectangle (in th case a square) whose sides have length 4 has an area of 16 square units and a perimeter of 16 unit Analogously, a rectangle that has side lengths 3 and 6 will have an area of 18 square units and perimeter of 18 units. These are the only two natural numbers where this property holds true.

Examining 16 individually, we find that it can be written as two base/powers interchanged as 16 $2^4$ and 16 = $4^2$. No other number has this property![7]

Recalling triangular numbers, the number 16 can be written as the sum of triangular numbers[8] two ways. It is the smallest such square number that has this characteristic. 16 = 6 + 10 = 1 + 15.

The Pythagoreans were enchanted with these two numbers and despised the number 17, whic separated them. However, the number 17 has some properties that give it importance as well. Th number 17 is the seventh prime number, and 17 generates the sixth Mersenne number,[9] 131,07 Furthermore, 17 is the sum of the first four prime numbers: 2 + 3 + 5 + 7 = 17. Perhaps its mo

famous manifestation is that it represents the number of sides of a regular polygon that can b
~~constructed with straightedge and compass—which the famous German mathematician Carl Friedric~~
Gauss (1777–1855) was very proud to have proved in his early years. He was so proud of th
discovery that it was ultimately constructed on his tombstone.

The number 17 also has some strange characteristics. For example, $17^3$ = 4,913 and the sum of th
digits of this number is: 4 + 9 + 1 + 3 = 17. By the way, the only other numbers that share th
characteristic are: 1, 8, 18, 26, and 27. You might want check this to be convinced of this property. W
offer one here: $26^3$ = 17,576, and 1 + 7 + 5 + 7 + 6 = 26.

Some prime numbers, when their digits are reversed, also deliver prime numbers. As you can se
from the list of the first few of these, the number 17 is one of these unusual numbers:

13, 17, 31, 37, 71, 73, 79, 97, 107, 113, 149, 157,…

Let us now inspect what can make the number 18 stand out.

We should first notice that similar to the number 17, the number 18, when spoken quickly, is ofte
confused with the number 80. In our society, the number 18 represents the number of holes on
complete golf course, the number of wheels on a trailer truck, and the minimum voting age in man
states.

Yet another curiosity can be seen with the two eighteen-letter words *conservationalists* an
*conversationalists* are anagrams of each other. They are the longest pair of anagrams in the Englis
language—if scientific words are excluded.[10]

From ancient times the number 18 has enjoyed a peculiar popularity among those who understan
the Hebrew language. For centuries, Hebrew scholars have used a procedure called *gematria*
analyze the scriptures. This technique involves having the letters of a word in the Hebrew languag
take on their numerical equivalent. When one does this with the number 18 as expressed with Hebre
characters, it looks like this חי. In the Hebrew language, when seen as a word, these two letters spe
out the word *life* and often it is seen as a good luck charm. In the Chinese culture, the number 1
represents a word that means that someone will prosper.

However, the number 18 also holds some interesting mathematical properties. For example, it is th
only number that is twice the sum of its digits. Yet, when we look at this phenomenon, we can exten
this unusual fact as follows: 18 = 9 + 9, and its reversal 81 = 9 · 9, which is the square of the sum of i
digits. We can continue this pattern by inserting a 9 between the two digits of 18 and get th
following: 198 = 99 + 99, and again reversing the digits: 891 = 9 · 99. While on the topic, we can se
that 18 + 81 = 99, and 9 + 9 = 18. However, we can extend this strange property as shown here.

$$\begin{array}{rcrrcr} 18 &=& 9+9 & 81 &=& 9\cdot 9 \\ 198 &=& 99+99 & 891 &=& 9\cdot 99 \\ 1998 &=& 999+999 & 8991 &=& 9\cdot 999 \\ 19998 &=& 9999+9999 & 89991 &=& 9\cdot 9999 \end{array}$$

…and it continues!

Another property held by the number 18 is seen when we take 18 to the third and fourth powers an
inspect the two products. We will find that each of the digits from 0 to 9 was used exactly once:

$18^3$ = 5,832, and $18^4$ = 104,976.

Taking this a step further, one may have noticed that the sum of the digits of $18^3$ = 5,832, whos
digit sum is 5 + 8 + 3 + 2 = 18. This, in and of itself, would be quite spectacular; however, this can b
even further extended, when we look at the sixth and seventh powers of 18. Once again, these powe
of 18 yield numbers where the sum of their digits is 18. That is, $18^6$ = 34,012,224, and 3 + 4 + 0 + 1

$2 + 2 + 2 + 4 = 18$; as well as $18^7 = 612{,}220{,}032$, and $6 + 1 + 2 + 2 + 2 + 0 + 0 + 3 + 2 = 18$.

Just to take this to a “higher” level, we find that

$18^{18} = 39{,}346{,}408{,}075{,}296{,}537{,}575{,}424$, and

$3 + 9 + 3 + 4 + 6 + 4 + 0 + 8 + 0 + 7 + 5 + 2 + 9 + 6 + 5 + 3 + 7 + 5 + 7 + 5 + 4 + 2 + 4 = 108$.

We can also have some fun with the number 18. Begin by taking any three-digit number whos digits are all different, and then arrange them in order to form the largest number and then th smallest number. Subtract the smaller number from the larger number, and you will find that th answer will be a number, the sum of whose digits is 18.

Let's consider an example. We shall select the number 584. We then write the smallest number wit these digits, 458, and then the largest number with these digits, 854. Subtracting these numbers, 854 458, we get 396. The sum of the digits of this number $(3 + 9 + 6)$ is 18. Try it with some other thre digit number to convince yourself of this property. This makes a great impression at any dinner par you attend!

We can also look at the eighteenth Fibonacci number[11] and show that it is equal to the sum of th cubes of four consecutive numbers, as shown here: $F_{18} = 2{,}584 = 7^3 + 8^3 + 9^3 + 10^3$.

You may recall that 18 is also the sixth Lucas number.[12] You might want to look for other suc surprise appearances of this popular number 18.

## THE CURIOUS NUMBER 30

The number 30 is the sum of the first four square numbers:

$1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$.

The number 30 is also the largest number where all the co-primes (numbers that are relative prime, i.e., their only common factor is 1) smaller than itself (except for 1) are prime numbers. The are: 7, 11, 13, 17, 19, 23, and 29. The other numbers that have this property are 3, 4, 6, 8, 12, 18, 24, s you can see that 30 is the largest number having this property.

In 1907, a German student, H. Bonse, developed an elementary proof[13]—without calculus—of th fact about 30. The proof can also be found in the book *The Enjoyment of Mathematics: Selections fro Mathematics for the Amateur*, by Rademacher and Toeplitz.[14]

## THE CURIOUS NUMBER 37

We know that the number 37 is a prime number, but it also has some rather-unique properties that w will consider here. Let us call the sum of the squares of the digits of a number $n$ as $Q^2(n)$. So for th number 37 we have $Q^2(37) = 3^2 + 7^2 = 58$. This, so far, is not very impressive, but if we now subtra the product of the digits from this number, we will get (surprisingly) $Q^2(37) - 3{\cdot}7 = 58 - 21 = 37$.

Or put another way, $Q^2(37) = 37 + 3 \cdot 7$. This a rather-amazing property, such that one gets wonder if there are other two-digit numbers that have this same property. In general terms, $n =$ where $a, b \in \{0, 1, 2, 3\ldots, 9\}$ and $a \neq 0$; keep in mind that $\overline{ab}$ is a two-digit number in the decim system, such that $Q^2(\overline{ab}) - a \cdot b = \overline{ab}$. Another way of expressing this is: $Q^2(\overline{ab}) = \overline{ab} + a \cdot b$.

To answer our question about there being numbers other than 37 that have this property, w consider the general case that appears as $a^2 + b^2 - a \cdot b = 10a + b$. With the aid of a computer we fi that there is only one other number that shares this property with 37, and that is 48. Since for $n = 4$