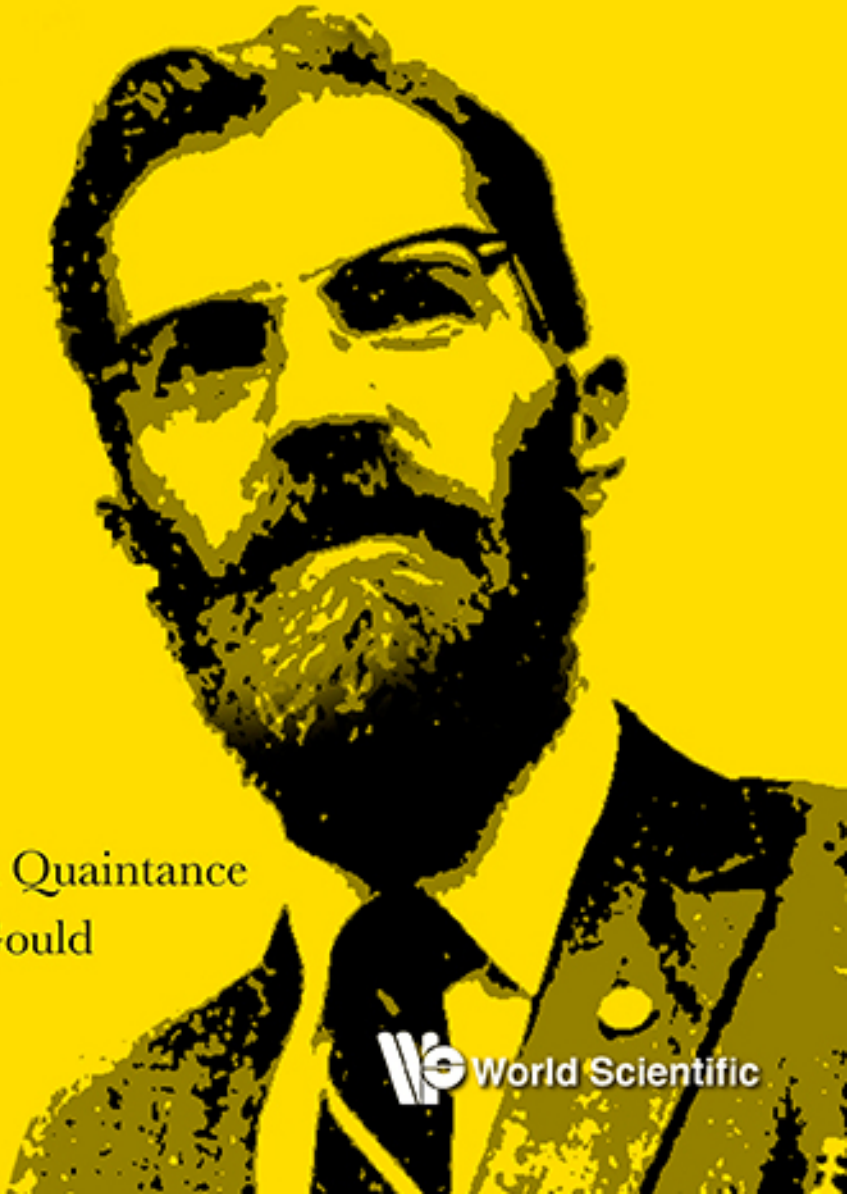
Combinatorial Identities for Stirling Numbers

The Unpublished Notes of H W Gould



Jocelyn Quaintance
H W Gould

World Scientific

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Jocelyn Quaintance
University of Pennsylvania, USA

H. W. Gould
West Virginia University, USA

World Scientific

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COMBINATORIAL IDENTITIES FOR STIRLING NUMBERS
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To my parents for all their patience, love, and support.
To Jean, fundraiser extraordinaire.

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Foreword

The book you hold in your hands is a uniquely valuable work revealing the tastes and interests of Henry Gould. It is a mixture of Henry's own personal development of the Stirling numbers, combined with an incredible historical knowledge of the subject. Indeed, this book reminds me in many ways of Nathan Fine's book, *Basic Hypergeometric Series and Applications* (for which I wrote the preface).

Henry Gould's first paper on Stirling numbers appeared in 1960 (*Proc. American Math. Soc.*, 11 (1960), 447-451). This first paper is typical of Gould's style and approach. The question addressed concerned formulas by Schlömilch and Schäfli which present representations of Stirling numbers of the first kind by Stirling numbers of the second kind. Gould asked, quite naturally, can we reverse this so that numbers of the second kind are represented by numbers of the first kind? The beautiful answer he provided is:

$$S_2(n-k,k) = \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} S_1(k+j-1,k).$$

In the interim, he has written many further papers on this topic. Indeed this book is, in some sense, an extensive expansion and completion of one aspect of class notes he prepared a number of years ago entitled *Sums of Powers of Numbers*.

Essential for this book is the historical understanding that Henry Gould brings to the subject. This historical expertise is perhaps most clearly revealed in a letter he wrote some time ago to Richard Askey. Henry never officially got a Ph.D. While this is uncommon today especially in the U.S., it was a common practice in England early in the 19th and early 20th century; in particular, G.H. Hardy never had a Ph.D. (he would have considered it

beneath him). However, Henry is actually an unofficial student of Leonard Carlitz. Here better than anything I can say are Henry's words describing his study of history and his debt to Carlitz:

> " I remember my first visit to Duke, circa 1953, before I got my B.A. degree (that came in 1954). It was at the suggestion of Leonard Carlitz. I had read some of his papers and wrote to him, sending him some formulas I had discovered. He responded that I should visit Duke and look at this book and that book and this journal and another. ... that some of my formulas were new while others were old. For example I had "rediscovered" Abel's extension of the binomial formula, etc. while I was in high school.
>
> Carlitz knew what was in most every paper in the library. I also used to have that ability early on. I used to spend whole days in the libraries at Virginia, Carolina, Duke, Library of Congress, and numerous other places, trying to find what was known. I began years ago jotting things down in notebooks, then on file cards, and now have over 30,000 3" by 5" file cards on the literature that interested me. When I first came here to WVU 38 years ago, I began going through Crelle's and Liouville's journals and the Jahrbuch ueber die Fortschritte der Mathematik, Zentralblatt, Math. Reviews and every journal we had in fact, page by page. I remember many nights when I would sleep in the library, on the hard floor sometimes when I got tired after making notes on 50 volumes of some journal.
>
> Consequently I can recall times when I would visit Carlitz or talk with him on the phone and mention some paper by (e.g.) Gegenbauer, and we would both know the paper and both remark that the author proved 300 formulas or whatever ... it was all at our mental fingertips. I often take a student to the library and just walk over and pull down a journal volume to show the student some specific reference and the student is always amazed that I remember exactly where to look!"

That, dear reader, is Henry Gould, a man deeply in love with mathematics, especially combinatorial mathematics. Not surprisingly, many of the 30,000 file cards are related to Stirling numbers. It is good to see that his insights and understanding are finally being published.

We owe Jocelyn Quaintance a debt for guiding the writing and persevering in making sure that this grand project finally made it into print.

George E. Andrews

Preface

This book is the result of a life changing relationship. In 2006 I was a recently graduated Ph.D. trying to publish my first combinatorial research papers with little success. In order to become a research mathematician, I realized that I had to learn what topics led to publication-worthy research projects. I thought the best way to gain these insights was to take a year off and work with an expert in the field of enumerative combinatorics. I asked George Andrews who he recommended I contact as a potential advisor. He suggested Henry Gould. I sent Henry an email not knowing what to expect. Much to my surprise and delight, Henry quickly replied that he would be most pleased to become my mentor. Therefore, I quit my job and in August 2006 travelled to Morgantown, WV to begin my studies with Henry. Henry introduced me to the wonderful world of binomial identities, special functions, Catalan numbers, and Stirling numbers. Under his guidance, I learned how to formulate research questions, the techniques needed to solve these questions, and how to present the results for publication. During the next four years, Henry and I collaborated on over ten papers. When working on these papers, Henry often referred to his "bible", a much used copy of his book entitled *Combinatorial Identities*, a collection of tables containing over 500 binomial identities. I asked Henry if he had the proofs for all of the identities in this book. He told me that his proofs existed in an unpublished manuscript of seven handwritten volumes which he started writing before he was an undergraduate at the University of Virginia. Henry said he was loathe to show anyone his private notebooks since they were a diary of his mathematical development over the past 50 years and were not in publication ready form. But since we had developed such a rapport, he would show me a few excerpts. When I saw the excerpts, I realized that I was looking at something special that needed to be shared with the math-

ematical community. I told Henry this and because he trusted me, he was willing to give me a copy of the 2100+ handwritten pages to prepare for publication. Thus began a three year odyssey, the end result is this book. The first eight chapters introduce readers to the special techniques Henry uses in proving his binomial identities. We do not assume the reader has any special mathematical background and have written this material in a style which should be readily accessible by anyone who has taken a year of calculus and an undergraduate level course in discrete mathematics. The second half of the book uses the techniques espoused in the first half and is geared towards the research mathematician. It focuses on connections between various kinds of Stirling numbers, a topic on which Henry is a world-renowned expert. As far as we know, Chapters 9 through 15 are the only source which systematically records Henry's unique results interrelating Stirling numbers of the first kind, Stirling numbers of the second kind, Worpitzky numbers, Bernoulli numbers, and Nörlund polynomials. Researchers should particularly focus on Chapters 13 through 15.

My fervent hope is this book will introduce a new generation of mathematicians to the world of combinatorial identities by teaching them the skills necessary for discovering new identities, just as Henry patiently taught me over the past ten years. That would be the most fitting tribute to Henry's legacy.

Jocelyn Quaintance

Acknowledgments

Henry and Jocelyn thank Dr. Addison Fischer for his three years of financial support. Jocelyn thanks Drs. Monique and Harry Gingold (and Beshy) for being gracious hosts. She also thanks João Sedoc for his help in typesetting the manuscript in the publisher's format.

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Chapter 1

Basic Properties of Series

The purpose of this book is to develop Professor Gould's formulas for relating Stirling numbers of the second kind to Stirling numbers of the first kind via Bernoulli numbers. Many of these relationships rely on Professor Gould's techniques for evaluating series whose summands are binomial coefficients. Therefore, the first eight chapters of this book will be a primer on these various techniques. Assume n and k are nonnegative integers with $k \leq n$. Define $n!$ to be the product of the first n positive integers and $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ with $0! := 1$. Combinatorially $\binom{n}{k}$ counts the number of subsets of size k made from a set with n elements. We say $\binom{n}{k}$ is a **binomial coefficient**. The binomial coefficients are often displayed in Pascal's triangle. Pascal's triangle begins with $\binom{0}{0}$, and for row i, with $i \geq 2$, the jth entry from the left is $\binom{i-1}{j-1}$. Each row of the Pascal's triangle contains only a finite number of entries since $\binom{n}{k} = 0$ whenever $k > n$.

							1							
						1		1						
					1		2		1					
				1		**3**		**3**		1				
			1		4		**6**		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Table 1.1: A portion of Pascal's triangle

There is an inductive way to construct the rows of Pascal's triangle which is known as Pascal's identity.

Pascal's Identity: Let n and k be nonnegative integers, with $0 \leq k \leq n$. Define $\binom{n}{k} = 0$ if k is a negative integer. Then

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \tag{1.1}$$

Here is a combinatorial proof of Pascal's identity. In this and other combinatorial arguments we show both sides of a given equation count the same quantity. Suppose $S = \{1, 2, 3, ..., n+1\}$. The left side of Equation (1.1) counts the number of subsets of S with size k. Call such a subset a k-subset of S. We claim the right side of Equation (1.1) also counts the k-subsets of S. There are two distinct types of k-subsets. The first type of k-subset contains $n+1$. To complete such a subset, we must choose $k-1$ elements from $\{1, 2, ..., n\}$ in $\binom{n}{k-1}$ possible ways. The second type of k-subset does not contain $n+1$, and we must select k elements from $\{1, 2, ..., n\}$ in $\binom{n}{k}$ possible. Adding the two possibilities together counts all the k-subsets of S without repetition and produces Equation (1.1).

Mathematicians often generalize definitions, and the binomial coefficients are no exceptions. The typical way to generalize $\binom{n}{k}$ is to let n be an arbitrary complex number. We will always assume, unless otherwise specified, that k is a nonnegative integer, and x is a complex number. Define $\binom{x}{0} := 1$ and

$$\binom{x}{k} = \frac{x(x-1)(x-2)...(x-k+1)}{k!}, \qquad \text{for complex } x. \tag{1.2}$$

We say $\binom{x}{k}$ is a **general binomial coefficient**. Whenever $x = n$, Equation (1.2) corresponds to the traditional combinatorial definition of a binomial coefficient.

It is important to note that $\binom{x}{k}$ is a polynomial in x of degree k. This observation, along the with Fundamental theorem of algebra, proves many binomial identities.

Equation (1.1) generalizes as follows:

Pascal's Identity for general binomial coefficients: Let x be a complex number and k a nonnegative integer. Define $\binom{x}{k} = 0$ if k is a negative integer. Then

$$\binom{x+1}{k} = \binom{x}{k} + \binom{x}{k-1}. \tag{1.3}$$

The proof of Equation (1.3) is a matter of applying the definition provided by Equation (1.2). In particular

$$\begin{aligned}\binom{x}{k}+\binom{x}{k-1} &= \frac{x(x-1)...(x-k+1)}{k!}+\frac{x(x-1)...(x-k+2)}{(k-1)!}\\ &= \frac{x(x-1)...(x-k+2)[(x-k+1)+k]}{k!}\\ &= \frac{(x+1)x(x-1)...(x-k+2)}{k!}=\binom{x+1}{k}.\end{aligned}$$

There are four other binomial identities necessary for understanding the material in the following chapters. All of these identities are proven by applications of Equation (1.2). We leave the proofs of the first three identities for the reader and provide a detailed proof of the $-\frac{1}{2}$-Transformation.

Committee/Chair Identity: Let x be a complex number and n be a nonnegative integer. Then

$$(n+1)\binom{x}{n+1}=x\binom{x-1}{n}. \tag{1.4}$$

Cancellation Identity: Assume x is a complex number, while n and k are nonnegative integers such that $n \geq k$. Then

$$\binom{x}{n}\binom{n}{k}=\binom{x-k}{n-k}\binom{x}{k}. \tag{1.5}$$

-1-Transformation: Let x be a complex number and k be a nonnegative integer. Then

$$\binom{x}{k}=(-1)^k\binom{-x+k-1}{k}. \tag{1.6}$$

$-\frac{1}{2}$-Transformation: Let n be a nonnegative integer. Then

$$\binom{-\frac{1}{2}}{n}=\frac{(-1)^n}{2^{2n}}\binom{2n}{n}. \tag{1.7}$$

Proof: Using Equation (1.2) we see that

$$\begin{aligned}\binom{-\frac{1}{2}}{n} &= \frac{(-\frac{1}{2})(-\frac{3}{2})...(-\frac{1}{2}-n+1)}{n!}=\frac{(-1)^n}{2^n}\cdot\frac{1\cdot 3\cdot ...\cdot(2n-1)}{n!}\\ &= \frac{(-1)^n}{2^n}\cdot\frac{1\cdot 3\cdot ...\cdot(2n-1)}{n!}\cdot\frac{2\cdot 4\cdot ...\cdot 2n}{2\cdot 4\cdot ...\cdot 2n}\\ &= \frac{(-1)^n}{2^n}\cdot\frac{(2n)!}{n!}\cdot\frac{1}{2^n(1\cdot 2\cdot ...\cdot n)}=\frac{(-1)^n}{2^{2n}}\cdot\frac{(2n)!}{n!n!}\\ &= \frac{(-1)^n}{2^{2n}}\binom{2n}{n}.\end{aligned}$$

□

Throughout the first eight chapters the primary object of study will be finite series $\sum_{k=0}^{n} a_k = a_1 + a_2 + \cdots + a_n$ where each a_k is either a real or complex number. The k is the index of the series and a_k is often a product of binomial coefficients. Occasionally we will work with infinite series $\sum_{n=0}^{\infty} a_n$. In most cases the infinite series will be of the form $\sum_{n=0}^{\infty} a_k x^k$, where x is a formal parameter. We interpret $\sum_{n=0}^{\infty} a_k x^k$ as a *formal power series*, ignore questions of convergence, and instead focus on the algebraic and combinatorial manipulations of series. For more information on formal powers series, we refer the reader to [Wilf, 2014, Chap.2].

1.1 General Considerations of $\sum_{k=a}^{n} f(k)$

We begin our study of algebraic manipulations for finite series by stating the following collection of basic identities which are implicitly used throughout Professor Gould's work. Since many of these identities are self evident, we leave the proofs to the reader. Without loss of generality we assume f and g are arbitrary functions whose domains are the set of all nonnegative integers and whose ranges are the set of complex numbers. We also assume, unless otherwise specified, that n and a are nonnegative integers.

Linearity Properties: Let c be a complex number not indexed by k. Then

$$\sum_{k=a}^{n} f(k) + \sum_{k=a}^{n} g(k) = \sum_{k=a}^{n} [f(k) + g(k)] \tag{1.8}$$

$$\sum_{k=a}^{n} cf(k) = c\sum_{k=a}^{n} f(k). \tag{1.9}$$

Identity 1: Let $0 < k < a - 1 \leq n$. Then

$$\sum_{k=0}^{n} f(k) = \sum_{k=0}^{a-1} f(k) + \sum_{k=a}^{n} f(k). \tag{1.10}$$

Identity 2: Let $0 < k_1 < k_2 < ... < k_{m-1} < n$.

$$\begin{aligned}\sum_{j=0}^{n} f(j) &= \sum_{j=0}^{k_1} f(j) + \sum_{j=k_1+1}^{k_2} f(j) + ... + \sum_{j=k_{m-2}+1}^{k_{m-1}} f(j) + \sum_{j=k_{m-1}+1}^{n} f(j) \\ &= \sum_{i=0}^{m-1} \sum_{j=k_i+1}^{k_{i+1}} f(j), \qquad \text{where } k_0 + 1 := 0 \text{ and } k_m = n. \end{aligned} \tag{1.11}$$

Identity 3: Let x be a real number. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let a and n be nonnegative integers such that $n \geq a+1$. Then

$$\sum_{k=a}^{n} f(k) + \sum_{k=a}^{n} (-1)^k f(k) = 2 \sum_{k=\lfloor \frac{a+1}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} f(2k). \tag{1.12}$$

Identity 4: Let x be a real number and let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let a and n be nonnegative integers such that $n \geq a+1$. Then

$$\sum_{k=a}^{n} f(k) - \sum_{k=a}^{n} (-1)^k f(k) = 2 \sum_{k=\lfloor \frac{a+2}{2} \rfloor}^{\lfloor \frac{n+1}{2} \rfloor} f(2k-1). \tag{1.13}$$

If we add Identity 3 to Identity 4 we obtain

Identity 5: (bisection of a series) Let $n \geq a+1$. Then

$$\begin{aligned} \sum_{k=a}^{n} f(k) &= \sum_{k=\lfloor \frac{a+1}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} f(2k) + \sum_{k=\lfloor \frac{a+2}{2} \rfloor}^{\lfloor \frac{n+1}{2} \rfloor} f(2k-1) \\ &= \sum_{k=\lfloor \frac{a+1}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} f(2k) + \sum_{k=\lfloor \frac{a}{2} \rfloor}^{\lfloor \frac{n-1}{2} \rfloor} f(2k+1). \end{aligned} \tag{1.14}$$

We now state the following generalization of Identity 5:

Identity 6: (multi-section of a series) Let r be a positive integer. Let n and a be nonnegative integers such that $n-a+1 \geq r$. Then

$$\sum_{k=a}^{n} f(k) = \sum_{j=0}^{r-1} \sum_{k=\lfloor \frac{a+r-1-j}{r} \rfloor}^{\lfloor \frac{n-j}{r} \rfloor} f(rk+j). \tag{1.15}$$

Identity 5 is the case when $r = 2$.

For a fixed r Identity 6 partitions the sum modulo remainder classes. The number of remainder classes is indexed by j.

In practice Identity 6 often occurs with $a = 0$ and the number of summands being a multiple of r, in which case Equation (1.15) is equivalent to

Identity 6a: (multi-section of a series) Let r be a positive integer. Then

$$\sum_{k=0}^{rn-1} f(k) = \sum_{j=0}^{r-1}\sum_{k=0}^{n-1} f(rk+j). \tag{1.16}$$

A useful special case of Identity 5 occurs when $f(k) \to (-1)^k f(k)$, in which case we have

Identity 7: (alternating bisection formula) Let n and a be nonnegative integers such that $n \geq a+1$. Then

$$\begin{aligned}\sum_{k=a}^{n}(-1)^k f(k) &= \sum_{k=\lfloor\frac{a+1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} f(2k) - \sum_{k=\lfloor\frac{a+2}{2}\rfloor}^{\lfloor\frac{n+1}{2}\rfloor} f(2k-1) \\ &= \sum_{k=\lfloor\frac{a+1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} f(2k) - \sum_{k=\lfloor\frac{a}{2}\rfloor}^{\lfloor\frac{n-1}{2}\rfloor} f(2k+1). \end{aligned} \tag{1.17}$$

A similar substitution in Identity 6 provides

Identity 8: (alternating multi-section formula) Let r be a positive integer. Let n and a be nonnegative integers such that $n-a+1 \geq r$. Then

$$\sum_{k=a}^{n}(-1)^k f(k) = \sum_{j=0}^{r-1}\sum_{k=\lfloor\frac{a+r-1-j}{r}\rfloor}^{\lfloor\frac{n-j}{r}\rfloor} (-1)^{rk+j} f(rk+j). \tag{1.18}$$

Identity 9: (basic telescoping formula) Let r and n be fixed positive integers. Then

$$\sum_{k=1}^{n}\left(f(k)-f(k+r)\right) = \sum_{k=1}^{r}\left(f(k)-f(k+n)\right). \tag{1.19}$$

Both sums of Equation (1.19) add and subtract elements from $S = \{f(1), ...f(n+r)\}$. The focus of Identity 9 is combining the $n+r$ elements of S in two different ways. On the left side we select $f(k)$, where $1 \leq k \leq n$, and for each selection subtract $f(k+r)$. On the right side we select $f(k)$, where $1 \leq k \leq r$, and then subtract $f(k+r)$. Identity 9 claims that these two different ways of telescopically combining various elements

from S provides the same sum. Clearly if $n = r$, Identity 9 is trivially true. So without loss of generality assume that $r < n$. Then

$$\begin{aligned}
\sum_{k=1}^{n} (f(k) - f(k+r)) &= \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n} f(k+r) \\
&= \sum_{k=1-r}^{k=n-r} f(k+r) - \sum_{k=1}^{n} f(k+r) \\
&= \sum_{k=1-r}^{0} f(k+r) - \sum_{k=n-r+1}^{n} f(k+r) \\
&= \sum_{k=1}^{r} f(k) - \sum_{k=1}^{r} f(k+n) \\
&= \sum_{k=1}^{r} (f(k) - f(k+n)) .
\end{aligned}$$

A useful interpretation of Identity 9 may be derived as follows. Rewrite Equation (1.19) as

$$r \sum_{k=1}^{n} \frac{f(k+r) - f(k)}{r} = n \sum_{k=1}^{r} \frac{f(k+n) - f(k)}{n}. \tag{1.20}$$

The summands of Equation (1.20) are commonly written in terms of the difference operator $\Delta_{k,p} := \frac{f(k+p)-f(k)}{p}$. Hence Equation (1.20) becomes

$$\frac{1}{n} \sum_{k=1}^{n} \Delta_{k,r} f(k) = \frac{1}{r} \sum_{k=1}^{r} \Delta_{k,n} f(k). \tag{1.21}$$

Equation (1.21) implies that for fixed positive integers n and r, the average of n differences in increments of r equals the average of r differences in increments of n.

Identity 10: (doubling formula) Let n be a fixed positive integer. Then

$$2 \sum_{k=1}^{n} f(k) = \sum_{k=1}^{2n} f\left(\left\lfloor \frac{k+1}{2} \right\rfloor \right). \tag{1.22}$$

An equivalent form of Identity 10 is obtained by setting $k \to k+1$ in the right hand sum.